Interior Paralinearization

Recall that after change of variables, $v(x,z) = \varphi(x, z+\sigma(x))$ satisfies

$$Ev =: (1+|\nabla\sigma|^2)\partial_z^2 v + \Delta v - 2\nabla\sigma\cdot\nabla\partial_z v - \partial_z v\Delta\sigma = 0, \quad z>0.$$

We now study the paralinearized version of this eqn:

Proposition. $\sigma\in H^s$, $v\in C^0([-1,0];H^s)$, $\partial_z v\in C^0([-1,0],H^{s-1})$ $(s\geq 3+d/2)$

Define the "good unknown":

$$u = v - T_{\partial_z v}\sigma,$$

then

$$Pu =: T_{(1+|\nabla\sigma|^2)}\partial_z^2 u - 2T_{\nabla\sigma}\cdot\nabla\partial_z u + \Delta u - T_{\Delta\sigma}\partial_z u$$

$$= f_0 \in C^0(H^{2s-3-d/2}).$$

That is, u satisfies the paralinearized version of the equation satisfied by v, up to an error in $H^{2s-3-d/2}$.

Pf. First note that

1). By Sobolev product rule $(a \in H^\alpha,\ b \in H^\beta \Rightarrow ab \in H^\gamma$ for $\gamma \le (\alpha \wedge \beta) \wedge (\alpha + \beta - d/2))$

- $\partial_t^2 v \in C^0(H^{s-2})$
- $\partial_t^3 v \in C^0(H^{s-3})$.
- $|\nabla \sigma|^2 \in H^{s-1}$.

2). By paralinearization lemma: $(a \in H^\alpha,\ b \in H^\beta \Rightarrow ab - T_a b - T_b a \in H^{\alpha+\beta-d/2})$

so e.g.

$$\nabla \sigma \cdot \nabla \partial_t v - T_{\nabla \sigma} \cdot \nabla \partial_t v - \nabla \sigma \cdot T_{\nabla \partial_t v} \in C^0(H^{s-1+s-2-d/2}) = C^0(H^{2s-3-d/2});$$

all such terms lie in $C^0(H^{2s-3-d/2})$

Collecting all such terms together, we see that

$$Ev - Pv + R := (1+|\nabla\sigma|^2)\partial_z^2 v - T_{(1+|\nabla\sigma|^2)}\partial_z^2 v - T_{\partial_z^2 v}(|\nabla\sigma|^2)$$

$$-2\left(\nabla\sigma\cdot\nabla\partial_z v - T_{\nabla\sigma}\cdot\nabla\partial_z v - T_{\nabla\partial_z v}\cdot\nabla\sigma\right)$$

$$-\left(\Delta\sigma\,\partial_z v - T_{\Delta\sigma}\,\partial_z v - T_{\partial_z v}\Delta\sigma\right)$$

$$\in C^0(H^{2s-3-d/2}).$$

with

$$R := -T_{\partial_z^2 v}(|\nabla\sigma|^2) + 2T_{\nabla\partial_z v}\cdot\nabla\sigma + T_{\partial_z v}\Delta\sigma.$$

Writing $v = u + T_{\partial_z v}\sigma$, we have

$$Ev - P(u + T_{\partial_z v}\sigma) + R \in C_0(H^{2s-3-d/2}).$$

Since $Ev = 0$, we need to show

$$-P(T_{\partial_z v}\sigma) + R \in C_0(H^{2s-3-d/2}).$$

From $P(T_{\partial_z v}\sigma)$, we have (since σ is independent of z)

$$T_{(1+|\nabla\sigma|^2)}\partial_z^2(T_{\partial_z v}\sigma) = T_{(1+|\nabla\sigma|^2)}T_{\partial_z^3 v}\sigma$$

$$-2T_{\nabla\sigma}\cdot\nabla\partial_z(T_{\partial_z v}\sigma) = -2T_{\nabla\sigma}\cdot\nabla T_{\partial_z^2 v}\sigma$$

$$\Delta(T_{\partial_z v}\sigma) = T_{\partial_z\Delta v}\sigma + T_{\partial_z v}\Delta\sigma + 2T_{\partial_z\nabla v}\cdot\nabla\sigma$$

$$-T_{\Delta\sigma}\partial_z(T_{\partial_z v}\sigma) = -T_{\Delta\sigma}T_{\partial_z^2 v}\sigma.$$

Cancelling appropriate terms, it remains to show that

$$(*)\quad T_{(1+|\nabla\sigma|^2)}T_{\partial^3 v}\sigma \underbrace{-2T_{\nabla\sigma}\cdot\nabla T_{\partial_z^2 v}\sigma}_{= T_{\nabla\sigma}\cdot T_{\nabla\partial_z^2 v}\sigma + T_{\nabla\sigma}\cdot T_{\partial_z^2 v}\nabla\sigma} + T_{\partial_z\Delta v}\sigma - T_{\Delta\sigma}T_{\partial_z^2 v}\sigma + T_{\partial_z^2 v}(|\nabla\sigma|^2) \in C_0(H^{2s-3-d/2}).$$

Now note that

$$(1+|\nabla\sigma|^2)\partial_z^3 v - 2\nabla\sigma\cdot\nabla\partial_z^2 v + \partial_z\Delta v - \Delta\sigma\,\partial_z^2 v$$

$$= \frac{\partial}{\partial z}Ev$$

$$= 0$$

So $T_{\frac{\partial}{\partial z}} E v$ is order ≤ 0.

2). By paralinearization lemma: ($a \in H^\alpha$, $b \in H^\beta$. $T_a T_b - T_{ab}$ order $\le -(\alpha \wedge \beta - d/2)$)

so e.g.

$$|\nabla \sigma|^2 \in H^{s-1}, \quad \partial_z^3 v \in H^{s-3}, \text{ so}$$

$$T_{(1+|\nabla\sigma|^2)} T_{\partial_z^3 v} \text{ order } \le -(s-3-d/2).$$

since $\sigma \in H^s$, $T_{(1+|\nabla\sigma|^2)} T_{\partial_z^3 v} \sigma \in C_0(H^{2s-3-d/2})$.

3). The only remaining term is $\underbrace{+T_{\partial_z^2 v}(|\nabla\sigma|^2)}_{\text{from } R} - 2 T_{\nabla\sigma} \cdot T_{\partial_z^2 v} \nabla\sigma$.

but as before we now write

$$T_{\partial_z^2 v}\big(\underbrace{|\nabla\sigma|^2 - T_{\nabla\sigma}\cdot\nabla\sigma - T_{\nabla\sigma}\cdot\nabla\sigma}_{\in H^{s-1+s-1-d/2} = H^{2s-2-d/2}}\big)$$

and $\partial_z^2 v \in H^{s-2} \subseteq L^\infty \Rightarrow$ $T_{\partial_z^2 v}$ order ≤ 0.

$\square$