Paralinearization of the DN Operator

Now we carry out full paralinearization of the DN operator for the WWE.
As mentioned earlier, the starting point is elliptic factorization.

<u>Lemma</u>. $\exists\, a, A \in \Gamma^1_{s-1-d/2}$ s.t.

$$(\partial_z - T_a)(\partial_z - T_A)u = f \in C^0(H^{2s-3-d/2}).$$

where $u = v - T_{\partial_z v}\sigma$ is the good unknown.

<u>Pf</u>. From interior paralinearization, have

$$T_{(1+|\nabla\sigma|^2)}\partial_z^2 u - 2T_{\nabla\sigma}\cdot\nabla\partial_z u + \Delta u - T_{\Delta\sigma}\partial_z u \in C^0(H^{2s-3-d/2}).$$

With $b = \frac{1}{1+|\nabla\sigma|^2} \in H^{s-1}$, rewrite (multiply by T_b).

$$(*)\quad \partial_z^2 u - 2T_{b\nabla\sigma}\cdot\nabla\partial_z u + T_b\Delta u - T_{b\Delta\sigma}\partial_z u \in C^0(H^{2s-3-d/2})$$

Since

$$(\partial_z - T_a)(\partial_z - T_A) = \partial_z^2 - T_{A+a}\partial_z + T_aT_A$$

$$\sim \partial_z^2 - T_{A+a}\partial_z + T_{a\#A},$$

error $\in C^0(H^{2s-3-d/2})$, do later

replacing (*) by corresponding symbols, etc. we need to solve

$$a \# A = -b|\xi|^2 + r(x,\xi) \qquad (\text{corresp. to } T_{b\Delta u},\ r \text{ is remainder})$$

$$a + A = 2b(i\nabla\sigma\cdot\xi) + b\Delta\sigma \qquad (\text{corresp. to } -2T_{b\nabla\sigma}\cdot\nabla\partial_z u - T_{b\Delta\sigma}\partial_z u)$$

We express a and A as series and solve recursively:

$$a(x,\xi) = \sum_{k\le 1} a_k(x,\xi), \qquad A(x,\xi) = \sum_{\ell\le 1} A_\ell(x,\xi)$$

when e.g. a_k is "degree" (order) k.

First set $$a_1 A_1 = -b|\xi|^2, \quad a_1 + A_1 = 2ib\nabla\sigma\cdot\xi$$

by taking

$$a_1(x,\xi) = ib\nabla\sigma\cdot\xi - \sqrt{b|\xi|^2 - (b\nabla\sigma\cdot\xi)^2}$$

$$A_1(x,\xi) = ib\nabla\sigma\cdot\xi + \sqrt{b|\xi|^2 - (b\nabla\sigma\cdot\xi)^2}$$

Note that

- Since $b \le 1$, $b|\nabla\sigma| \le 1$, $b|\xi|^2 - (b\nabla\sigma\cdot\xi)^2 \ge b^2|\xi|^2$,
  so a_1, A_1 are well-defined.
- $a_1, A_1 \in H^{s-1}$, so by Sobolev embedding, $a_1, A \in \Gamma^1_{s-1-d/2}$

Recall that

$$a \# A = \sum_\alpha \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a_k \, \partial_x^\alpha A_\ell.$$

so the next order (0^{th}) term is to solve

$$a_0 A_1 + a_1 A_0 + \frac{1}{i}\partial_\xi a_1 \partial_x A_1 = 0$$

$$a_0 + A_0 = b\Delta\sigma$$

giving

$$a_0 = \frac{i\partial_\xi a_1 \cdot \partial_x A_1 - b\Delta\sigma a_1}{A_1 - a_1}, \quad A_0 = \frac{i\partial_\xi a_1 \cdot \partial_x A_1 - b\Delta\sigma A_1}{a_1 - A_1}.$$

Further terms are given by induction: kill off all

$a_{-m} + A_{-m}$ and lower $a_k \# A_\ell$ terms, giving:

$$A_{m-1} = -a_{m-1}, \quad a_{m-1} = \frac{1}{a_1 - A_1} \sum_{(k,\ell,\alpha)} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a_k \, \partial_x^\alpha A_\ell$$

$$m \le k \le 1, \quad m \le \ell \le 1, \quad |\alpha| = k + \ell - m.$$

To finish note that ~~y~~ since

- $v \in C_0(H^s)$, $\partial_z v \in C^0(H^{s-1}) \Rightarrow T_{\partial_z v}$ order ≤ 0

so $u = v - T_{\partial_z v} u \in C^0(H^s)$.

- the error $T_a T_A - T_{a \# A}$ is order
$$\leq 1 + 1 - (s - 1 - d/2) = 3 - s + d/2$$

and hence $(T_a T_A - T_{a \# A}) u \in C^0(H^{2s-3-d/2})$. $\square$

$s - (3 - s + d/2) = 2s - 3 - d/2$

We have $f = (\partial_z - T_a)(\partial_z - T_A) u \in C^0(H^{2s-3-d/2})$

Define $w = \partial_z - T_A$, write $a = a_1 + \tilde{a}$

Then w satisfies the heat eqn: (backwards):
$$\partial_z w - T_{a_1} w = T_{\tilde{a}} w + f.$$

By regularity result for heat eqn, we get

Cor. $w(0) = (\partial_z u - T_A u)|_{z=0} \in H^{2s-2-d/2-\varepsilon}$

for given $\varepsilon > 0$.

We can now collect together our results to obtain:

Thm. Let $d \geq 1$. $s \geq 3 + d/2$. $s - d/2 \notin \mathbb{N}$. and $\sigma \in H^s$. $v \in C^0([-1,0], H^s)$, $\partial_z v \in C^0(H^{s-1})$.

Then
$$G(\sigma)\psi = T_{\lambda_\sigma}(\psi - T_B \sigma) - T_V \cdot \nabla\sigma - T_{\nabla\cdot V}\sigma + R(\sigma, \psi)$$

where
$$\lambda_\sigma = (1+|\nabla\sigma|^2)A - i\nabla\sigma\cdot\xi, \quad \lambda_\sigma^1 = \sqrt{(1+|\nabla\sigma|^2)|\xi|^2 - (\nabla\sigma\cdot\xi)^2};$$
$$B := \partial_y \varphi = \frac{\nabla\sigma\cdot\nabla\psi + G(\sigma)\psi}{1+|\nabla\sigma|^2};$$
$$V := \partial_x \varphi = \nabla\psi - B\nabla\sigma;$$

Pf. • Recall that with the change of variables
$$v(x,z) = \varphi(x, z+\sigma(x))$$

we have
$$G(\sigma)\psi = (1+|\nabla\sigma|^2)\partial_z v - \nabla\sigma\cdot\nabla v\,\big|_{z=0} : Ev$$

- As before, we find that,

$$Ev = T_{(1+|\nabla\sigma|^2)}\partial_z v + 2T_{\partial_z v \nabla\sigma}\cdot\nabla\sigma - T_{\nabla\sigma}\cdot\nabla v - T_{\nabla v}\cdot\nabla\sigma + R,$$

with $R \in H^{s-1+s-1-d/2} = H^{2s-2-d/2}$.

Translating this to $u = v - T_{\partial_z v}\sigma$, we get

$$Ev = T_{(1+|\nabla\sigma|^2)}\partial_z u - T_{\nabla\sigma}\cdot\nabla u - T_{\nabla v - \partial_z v\nabla\sigma}\cdot\nabla\sigma + T_\alpha\sigma + R'$$

with $R' \in H^{2s-2-d/2}$ and

$$\alpha = (1+|\nabla\sigma|^2)\partial_z^2 u - \nabla\sigma\cdot\nabla\partial_z u.$$

The interior eqn implies

$$\alpha = -\nabla\cdot(\nabla v - \partial_z v\nabla\sigma),$$

so

$$Ev = T_{(1+|\nabla\sigma|^2)}\partial_z u - T_{\nabla\sigma}\cdot\nabla u - T_{\nabla v - \partial_z v\nabla\sigma}\cdot\nabla\sigma - T_{\nabla\cdot(\nabla v - \partial_z v\nabla\sigma)}\sigma + R' \quad (+)$$

- Now recall we had

$$(\partial_z - A)u|_{z=0} = R \in H^{2s-2-d/2-\varepsilon}$$

with

$$A_1 = \frac{i}{1+|\nabla\sigma|^2}\nabla\sigma\cdot\xi + \sqrt{\frac{|\xi|^2}{1+|\nabla\sigma|^2} - \left(\frac{\nabla\sigma\cdot\xi}{1+|\nabla\sigma|^2}\right)^2}$$

Thus,

$$T_{(1+|\nabla\sigma|^2)}\,\partial_z u - T_{\nabla\sigma}\cdot\nabla u = T_{\lambda_\sigma} u + R$$

with $$\lambda_\sigma = (1+|\nabla\sigma|^2)A - i\nabla\sigma\cdot\xi.$$

Hence $$\lambda_\sigma^{(1)} = \sqrt{(1+|\nabla\sigma|^2)|\xi|^2 - (\nabla\sigma\cdot\xi)^2}\,.$$

- Recalling the change of variables $v(x,t) = \varphi(x, t+\sigma(x))$,

$$\partial_z v = \partial_y \varphi = \mathcal{B}$$

$$-(\partial_z v)\nabla\sigma + \nabla_x v = \partial_x \varphi = \mathcal{V}$$

we have the desired result by $(+)$. $\square$