Here are collected some additional facts that come up in the solutions of some of the other qual problems. The references to the problems are given in the format $[X, \#\#\#\#, Y, \#]$ ([season, year, section, number]), where $X \in \{W, S, F\}$ (corresponding to winter, spring, fall) and $Y \in \{G, R, F, L\}$ (corresponding to groups, rings, fields, linear algebra).

1 Groups

- $[S, 2000, G, 2]$ Two elements of S_n are conjugate in S_n if and only if they have the same cycle type (write as disjoint cycles, the meaning of cycle type should be obvious. Then notice that if $(a_1 a_2 \ldots a_k)$ is a cycle, then $\tau(a_1 a_2 \ldots a_k)\tau^{-1} = (\tau(a_1)\tau(a_2) \ldots \tau(a_3)).$). So the number of conjugacy classes equals the number of partitions of n.

- $[S, 2000, G, 2]$ Let $m \leq n$ be positive integers. The number of m cycles in S_n is

$$\frac{\binom{n}{m} m!}{m} = \binom{n}{m}(m-1)! = \frac{n(n-1)(n-1)\ldots(n-m+1)}{m}.$$

- $[F, 2004, G, 1]$ Let G be a group and $Z(G)$ be the center of G.

$$G/Z(G) \text{ cyclic} \Longrightarrow G \text{ abelian}.$$

2 Rings

- $[F, 2004, R, 2]$ Euclid's Lemma: Let all unknowns be integers.

$$n \mid ab \text{ and } \gcd(n, a) = 1 \Longrightarrow n \mid b.$$

  If p is a prime, this says

$$p \mid ab \Longrightarrow p \mid a \text{ or } p \mid b.$$

3 Fields

- $[F, 2004, F, 2]$ Composite Extension: Let K/F be Galois and L/F is any extension. Then KL/L is Galois and

$$\mathrm{Gal}(KL/L) \cong \mathrm{Gal}(K/(K \cap L)).$$

  (Consider the map $\phi : \mathrm{Gal}(KL/L) \to \mathrm{Gal}(K/F) : \sigma \mapsto \sigma|_K$ and show ϕ is 1-1 and its image is $\mathrm{Gal}(K/(K \cap L))$: i.e. prove the fixed field of the image of ϕ is exactly $K \cap L$).

- $[F, 2004, F, 3]$ Primitive Element Theorem: Let K/F be a finite extension. Then $K = f(\theta)$ if and only if there exists only finitely many subfields of K containing F. In particular if K/F is finite and separable, then K/F is simple. (For the first statement, let E be an intermediate field and consider the minimal polynomial for θ over E, $g(x)$, and show E is generated over F by the coefficients of g. Conversely, divide into case of F finite or infinite. The finite case is immediate, the infinite case follows by considering subfields of the form $F(\alpha + c\beta)$ and showing that $F(\alpha, \beta) = F(\alpha + c\beta)$ for some $c \in F$. For the last statement, take the Galois closure of K over F.)

4 Linear Algebra

- $[S, 2004, L, 2]$ Similar polynomials have the same trace and determinant. So the trace and determinant is equal to the sum and product of the eigenvalues, respectively.

- $[F, 2004, L, 1]$ Let $A \in GL_n(\mathbf{C})$. Then A is similar to A^t (use Jordan Canonical Form and reverse order of basis).

- $[F, 2004, L, 1]$ Let $A, B \in Gl_n(\mathbf{C})$. Then

$$(AB)^t = B^t A^t.$$

(To see this, let $A : V \to W, B : W \to Z$ be linear transformations. Define $T : Z^* \to W^* : g \mapsto Tg : Tg(\alpha) = g(B(\alpha))$ and similarly define $S : W^* \to V^*$ (here V^* denotes the dual space of V, i.e. the space of linear functions from V into the base field $\mathbf{C}$). Explicit computation then shows that the matrix representation for T and S are A^t and B^t, repectively, from which the formula follows.) In particular, from this we see that the inverse operation and the transpose operation commute, i.e.

$$(P^{-1})^t = (P^t)^{-1}.$$

- $[F, 2004, L, 3]$ Let A be a symmetric matrix with real entries. Then A is diagonalizable.