Differential Geometry I: Einstein Summation Notation

Helen K. Lei

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Contents

Chapter 1

Notation

∘ We have some manifold M and fix some frame $\{\partial_\alpha\}$ and the corresponding dual frame $\{\theta^\gamma\}$ (so that $\theta^\gamma(\partial_\alpha) = \delta_\alpha^\beta$, where $\delta_\alpha^\beta = 1$ if $\alpha = \beta$ and 0 otherwise).

∘ The tangent space at the point $p \in \mathcal{M}$ is denoted T_pM and $\Gamma(M)$ denotes the space of vector fields.

∘ We will use Einstein summation notation, i.e., repeated indices (one upper and one lower) is summed. By convention, *covariant* indices (e.g., corresponding to tangent basis element or components of dual vectors) are below whereas *contravariant* indices (e.g., components of tangent vectors or dual basis elements) are above.

∘ We will use D_α (or ∇_α) to denote the covariant derivative in the α^{th} coordinate direction whereas ∂_α denotes directional derivative.

∘ Given a (semi–)Riemannian metric g, $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$, whereas $g^{\alpha\beta}$ denotes the corresponding entry in the inverse matrix to g, i.e., $g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$.

∘ In many places (especially in the beginning) we are loosely following parts of [1].

Chapter 2

Connection: $D_X Y$, ∇Y

We start with the definition of a *connection*, which is a mapping

$$D : T_P(M) \times \Gamma(M) \to T_P(M)$$

satisfying bilinearity and Leibnitz rule:

$$D_X(fY) = X(f)Y + fD_XY.$$

Here $X(f)$ is the usual directional derivative:

$$X(f) = df(X).$$

Writing $X = X^\alpha \partial_\alpha$, $Y = Y^\beta \partial_\beta$ and expanding, we see that

$$D_XY = X^\alpha[(\partial_\alpha Y^\beta)\partial_\beta + Y^\beta(D_{\partial_\alpha}\partial_\beta)],$$

which leads to the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ such that

$$D_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma$$

so that

$$D_XY = X^\alpha[(\partial_\alpha Y^\beta)\partial_\beta + Y^\beta\Gamma^\gamma_{\alpha\beta}\partial_\gamma].$$

Picking out the ∂_γ component $(D_XY)^\gamma$, we have

$$(D_XY)^\gamma = X^\alpha[\partial_\alpha Y^\gamma + Y^\beta\Gamma^\gamma_{\alpha\beta}]$$

Setting $X = \partial_\alpha$ and abbreviating $D_\alpha = D_{\partial_\alpha}$, we have

$$(D_\alpha Y)^\gamma = \partial_\alpha Y^\gamma + Y^\beta\Gamma^\gamma_{\alpha\beta}.$$

We *define* ∇Y to be the $(1,1)$–tensor given by the matrix coefficients $(D_\alpha Y)^\gamma$, i.e.,

$$\nabla Y = (D_\alpha Y)^\gamma(\theta^\alpha \otimes \partial_\gamma)$$

where $\{\theta^\alpha\}$ is the dual frame. Notice that ∇Y satisfies

$$\nabla Y(X, f) = (D_XY)(f)$$

Chapter 3

Torsion:

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \quad T^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}$$

Given the definition of the connection, we can now define a host of objects, starting with the torsion, which is a mapping

$$T : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$$

defined by

$$T(X,Y) = D_X Y - D_Y X - [X,Y].$$

The commutator $[X,Y]$ is interpreted via its action on functions $f \in C^{\infty}(M)$: Using Leibnitz rule, we see that we have e.g. (again with $X = X^{\alpha}\partial_{\alpha}\ Y = Y^{\beta}\partial_{\beta}$),

$$XY(f) = X^{\alpha}\partial_{\alpha}(Y(f)) = X^{\alpha}\partial_{\alpha}(Y^{\beta}\partial_{\beta}f) = X^{\alpha}((\partial_{\alpha}Y^{\beta})\partial_{\beta}f + Y^{\beta}\partial_{\alpha\beta}f)$$

(Here $\partial_{\alpha\beta} = \partial_{\alpha}\partial_{\beta} = \partial_{\beta\alpha}$.) We thus have

$$[T(X,Y)]^{\gamma} = [X^{\alpha}(D_{\alpha}Y)^{\gamma} - Y^{\beta}(D_{\beta}X)^{\gamma}] - [X^{\alpha}\partial_{\alpha}Y^{\gamma} - Y^{\beta}\partial_{\beta}X^{\gamma}].$$

Now we have e.g., $(D_{\alpha}Y)^{\gamma} = \partial_{\alpha}Y^{\gamma} + Y^{\beta}\Gamma^{\gamma}_{\alpha\beta}$ (the first term on the right cancels the corresponding term from XY) we see that

$$\begin{aligned} [T(X,Y)]^{\gamma} &= X^{\alpha}Y^{\beta}\Gamma^{\gamma}_{\alpha\beta} - X^{\alpha}Y^{\beta}\Gamma^{\gamma}_{\beta\alpha} \\ &= X^{\alpha}Y^{\beta}(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}) \\ &:= T^{\gamma}_{\alpha\beta}X^{\alpha}Y^{\beta}. \end{aligned}$$

Note that the torsion does not depend on the first derivatives of the *components* of X and Y (these terms canceled out).

Therefore, as before, we can define a (2, 1)–tensor by the "matrix" coefficients ($T^{\gamma}_{\alpha\beta}$):

$$T(X,Y)f := T^{\gamma}_{\alpha\beta}(\theta^{\alpha} \otimes \theta^{\beta} \otimes \partial_{\gamma})(X,Y,f).$$

Chapter 4

Curvature:

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}, \quad R^{\lambda}_{\gamma\alpha\beta} = \partial_\alpha \Gamma^{\lambda}_{\beta\gamma} - \partial_\beta \Gamma^{\lambda}_{\alpha\gamma} + \Gamma^{\mu}_{\beta\gamma}\Gamma^{\lambda}_{\alpha\mu} - \Gamma^{\mu}_{\alpha\gamma}\Gamma^{\lambda}_{\beta\mu}$$

The curvature is a mapping

$$R : \Gamma(M) \times \Gamma(M) \to \mathrm{Hom}(\Gamma, \Gamma)$$

defined by

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

Let us write $X = X^\alpha \partial_\alpha, Y = Y^\beta \partial_\beta, Z = Z^\gamma \partial_\gamma$. First, we have as before that (assuming $\partial_{\alpha\beta} = \partial_{\beta\alpha}$)

$$[X,Y]^\lambda = X^\alpha(\partial_\alpha Y^\lambda) - Y^\beta(\partial_\beta X^\lambda)$$

So

$$\begin{aligned} D_{[X,Y]}Z &= [X,Y]^\lambda((\partial_\lambda Z^\gamma)\partial_\gamma + Z^\gamma D_\lambda \partial_\gamma) \\ &= [X,Y]^\lambda((\partial_\lambda Z^\gamma)\partial_\gamma + Z^\gamma \Gamma^{\mu}_{\lambda\gamma}\partial_\mu) \\ &= [X,Y]^\lambda((\partial_\lambda Z^\mu) + Z^\gamma \Gamma^{\mu}_{\lambda\gamma})\partial_\mu \end{aligned}$$

On the other hand,

$$\begin{aligned} D_X D_Y Z &= X^\alpha D_\alpha(Y^\beta D_\beta Z) \\ &= X^\alpha[(\partial_\alpha Y^\beta) D_\beta Z + Y^\beta D_\alpha D_\beta Z] \\ &= X^\alpha(\partial_\alpha Y^\beta)(\partial_\beta Z^\lambda + Z^\gamma \Gamma^{\lambda}_{\beta\gamma})\partial_\lambda + X^\alpha Y^\beta D_\alpha D_\beta Z \\ &= X^\alpha(\partial_\alpha Y^\lambda)(\partial_\lambda Z^\mu + Z^\gamma \Gamma^{\mu}_{\lambda\gamma})\partial_\mu + X^\alpha Y^\lambda D_\alpha D_\lambda Z, \end{aligned}$$

where in the last line we have performed a change of indices $\beta \rightsquigarrow \lambda$ and $\lambda \rightsquigarrow \mu$, from which it is clear that the first term matches the contribution from the XY part of $D_{[X,Y]}Z$. Similarly, the corresponding term from the YX part of $D_{[X,Y]}Z$ matches the corresponding term from $D_Y D_X Z$. I.e., we have observed that

$$D_X D_Y Z = D_{[X,Y]}Z + X^\alpha Y^\beta (D_\alpha D_\beta - D_\beta D_\alpha)Z$$

or

$$R(X,Y)Z = X^\alpha Y^\beta (D_\alpha D_\beta - D_\beta D_\alpha)Z.$$

Next we have

$$\begin{aligned}
D_\alpha D_\beta Z &= D_\alpha[(\partial_\beta Z^\gamma)\partial_\gamma + Z^\gamma D_\beta \partial_\gamma] \\
&= D_\alpha[((\partial_\beta Z^\mu) + Z^\gamma \Gamma^\mu_{\beta\gamma})\partial_\mu] \\
&= [\partial_{\alpha\beta} Z^\mu + \partial_\alpha(Z^\gamma \Gamma^\mu_{\beta\gamma})]\partial_\mu + [(\partial_\beta Z^\mu) + Z^\gamma \Gamma^\mu_{\beta\gamma}]\Gamma^\lambda_{\alpha\mu}\partial_\lambda \\
&= [\partial_{\alpha\beta} Z^\lambda + \partial_\alpha(Z^\gamma \Gamma^\lambda_{\beta\gamma}) + ((\partial_\beta Z^\mu) + Z^\gamma \Gamma^\mu_{\beta\gamma})\Gamma^\lambda_{\alpha\mu}]\partial_\lambda \\
&= [\partial_{\alpha\beta} Z^\lambda + (\partial_\alpha Z^\gamma)\Gamma^\lambda_{\beta\gamma}) + Z^\gamma(\partial_\alpha \Gamma^\lambda_{\beta\gamma}) + ((\partial_\beta Z^\mu) + Z^\gamma \Gamma^\mu_{\beta\gamma})\Gamma^\lambda_{\alpha\mu}]\partial_\lambda
\end{aligned}$$

Interchanging α and β, we find that

$$D_\beta D_\alpha Z = [\partial_{\beta\alpha} Z^\lambda + (\partial_\beta Z^\gamma)\Gamma^\lambda_{\alpha\gamma} + Z^\gamma(\partial_\beta \Gamma^\lambda_{\alpha\gamma}) + ((\partial_\alpha Z^\mu) + Z^\gamma \Gamma^\mu_{\alpha\gamma})\Gamma^\lambda_{\beta\mu}]\partial_\lambda.$$

First we can cancel the $\partial_{\alpha\beta}$ and $\partial_{\beta\alpha}$ terms. Next observe that 1) $(\partial_\alpha Z^\gamma)\Gamma^\lambda_{\beta\gamma}$ matches up with $(\partial_\alpha Z^\mu)\Gamma^\lambda_{\beta\mu}$ and 2) $(\partial_\beta Z^\mu)\Gamma^\lambda_{\alpha\mu}$ matches up with $(\partial_\beta Z^\gamma)\Gamma^\lambda_{\alpha\gamma}$ if we rename $\gamma \rightsquigarrow \mu$. Thus all terms involving derivatives of Z^γ will be gone, and we are left with

$$[R(X,Y)Z]^\lambda = X^\alpha Y^\beta Z^\gamma \left[\partial_\alpha \Gamma^\lambda_{\beta\gamma} - \partial_\beta \Gamma^\lambda_{\alpha\gamma} + \Gamma^\mu_{\beta\gamma}\Gamma^\lambda_{\alpha\mu} - \Gamma^\mu_{\alpha\gamma}\Gamma^\lambda_{\beta\mu}\right] := X^\alpha Y^\beta Z^\gamma R^\lambda_{\gamma\alpha\beta}.$$

I.e., $R^\lambda_{\gamma\alpha\beta}$ is the λ–component of $R(\partial_\alpha, \partial_\beta)\partial_\gamma$.

Finally, in this case we have a (3, 1)–tensor

$$[R(X,Y)Z]f := R^\lambda_{\gamma\alpha\beta}(\theta^\gamma \otimes \theta^\alpha \otimes \theta^\beta \otimes \partial_\lambda)(Z,X,Y,f).$$

Chapter 5

Covariant Derivatives: (1, 0), (1, 1)–Tensors

For a function, the covariant derivative corresponds to $D_X(f) = X(f)$, the usual directional derivative, and for vector fields $D_X Y$ is the connection. We can now generalize the notion of covariant derivatives to higher order tensors in the "natural" way, satisfying the following properties:

- D_X preserves the type of tensor.
- D_X commutes with contraction, in the sense that e.g., if ω is a 1–form and Y is a vector field, then
$$D_X[\omega(Y)] = (D_X\omega)(Y) + \omega(D_X(Y)).$$
- D_X satisfies the Leibnitz rule:
$$D_X(u \otimes v) = (D_X u) \otimes v + u \otimes (D_X v).$$

Let us now compute the covariant derivative of a 1–form $\omega = \omega_\gamma \theta^\gamma$. Then the Leibnitz rule will allow us to compute the covariant derivative of general (r, s)–tensors by induction. Let us first consider $X = X^\alpha \partial_\alpha$ and $Y = Y^\beta \partial_\beta$ to show that $D_X\omega$ is indeed tensorial. It is trivial to contract ω with Y:

$$\omega(Y) = \omega_\gamma \theta^\gamma (Y^\beta \partial_\beta) = \omega_\gamma Y^\beta \delta^\gamma_\beta = \omega_\gamma Y^\gamma.$$

We therefore have

$$D_X(\omega_\gamma Y^\gamma) = D_X(\omega(Y)) = (D_X\omega)(Y) + \omega(D_X(Y)).$$

We will therefore obtain an expression for $D_X(\omega)(Y)$ if we compute $D_X(\omega_\gamma Y^\gamma)$ and $\omega(D_X(Y))$. The resulting expression will show that $D_X(\omega)$ is a tensor, and then the appropriate coordinate expression will emerge by considering $X = \partial_\alpha, Y = \partial_\beta$.

First, $\omega_\gamma Y^\gamma$ is a function, so

$$D_X(\omega_\gamma Y^\gamma) = X^\alpha \partial_\alpha(\omega_\gamma Y^\gamma) = X^\alpha[(\partial_\alpha \omega_\gamma)Y^\gamma + \omega_\gamma(\partial_\alpha Y^\gamma)].$$

On the other hand, $D_X(Y)$ is the connection, so as before, we obtain

$$D_X(Y) = X^\alpha[(\partial_\alpha Y^\lambda) + Y^\beta \Gamma^\lambda_{\alpha\beta}]\partial_\lambda.$$

So we have

$$\omega(D_X(Y)) = \omega_\gamma X^\alpha[(\partial_\alpha Y^\gamma) + Y^\beta\Gamma^\gamma_{\alpha\beta}].$$

Therefore

$$\begin{aligned}(D_X\omega)(Y) &= D_X(\omega_\gamma Y^\gamma) - \omega(D_X(Y))\\ &= X^\alpha[(\partial_\alpha\omega_\gamma)Y^\gamma + \omega_\gamma(\partial_\alpha Y^\gamma)] - \omega_\gamma X^\alpha[(\partial_\alpha Y^\gamma) + Y^\beta\Gamma^\gamma_{\alpha\beta}]\\ &= X^\alpha[(\partial_\alpha\omega_\gamma)Y^\gamma - \omega_\gamma Y^\beta\Gamma^\gamma_{\alpha\beta}]\\ &= X^\alpha Y^\beta[\partial_\alpha\omega_\beta - \omega_\gamma\Gamma^\gamma_{\alpha\beta}]\\ &:= X^\alpha Y^\beta(D_\alpha\omega)_\beta.\end{aligned}$$

I.e.,

$$D_X\omega = X^\alpha D_\alpha\omega = [X^\alpha(D_\alpha\omega)_\beta]\theta^\beta.$$

Thus $D_X\omega(Y)$ is indeed tensorial, and we have the (2, 0)–tensor $\nabla\omega$:

$$\nabla\omega(X,Y) := (D_\alpha\omega)_\beta(\theta^\alpha\otimes\theta^\beta)(X,Y).$$

For comparison purposes, let us now write down the covariant derivatives of a (1, 0)–tensor and (0, 1)–tensor:

$$\begin{aligned}(D_\alpha Y)^\lambda &= \partial_\alpha Y^\lambda + Y^\beta\Gamma^\lambda_{\alpha\beta}\\ (D_\alpha\omega)_\lambda &= \partial_\alpha\omega_\lambda - \omega_\gamma\Gamma^\gamma_{\alpha\lambda}\end{aligned}\tag{5.1}$$

So from the Leibnitz rule, we see that e.g., the covariant derivative of $v\otimes u$ a (1, 1)–tensor would be (with $u = u^\mu\partial_\mu, v = v_\mu\theta^\mu$)

$$\begin{aligned}D_\alpha(v\otimes u) &= (\partial_\alpha u^\lambda + u^\beta\Gamma^\lambda_{\alpha\beta})(v\otimes\partial_\lambda) + (\theta^\lambda\otimes u)(\partial_\alpha v_\lambda - v_\gamma\Gamma^\gamma_{\alpha\lambda})\\ &= v_\mu(\partial_\alpha u^\lambda + u^\beta\Gamma^\lambda_{\alpha\beta})(\theta^\mu\otimes\partial_\lambda) + (\theta^\lambda\otimes\partial_\mu)(\partial_\alpha v_\lambda - v_\gamma\Gamma^\gamma_{\alpha\lambda})u^\mu\\ &= v_\mu(\partial_\alpha u^\lambda + u^\beta\Gamma^\lambda_{\alpha\beta})(\theta^\mu\otimes\partial_\lambda) + (\theta^\mu\otimes\partial_\lambda)(\partial_\alpha v_\mu - v_\gamma\Gamma^\gamma_{\alpha\mu})u^\lambda\\ &= [v_\mu(\partial_\alpha u^\lambda + u^\beta\Gamma^\lambda_{\alpha\beta}) + (\partial_\alpha v_\mu - v_\gamma\Gamma^\gamma_{\alpha\mu})u^\lambda](\theta^\mu\otimes\partial_\lambda)\\ &= [D_\alpha(v\otimes u)]^\lambda_\mu\ (\theta^\mu\otimes\partial_\lambda),\end{aligned}$$

and thus, we have a $(2,1)$–tensor $\nabla(u\otimes v)$:

$$[\nabla(u\otimes v)](X,Y,f) := [D_\alpha(u\otimes v)]^\lambda_\mu\ (\theta^\alpha\otimes\theta^\mu\otimes\partial_\lambda)(X,Y,f).$$

Remark. Another way to compute the covariant derivative would be to first write e.g.,

$$v\otimes u = ((h^\mu_\lambda\theta^\lambda)\otimes\partial_\mu)$$

and then apply the Leibnitz rule. Indeed, it is with this way of computing that we can see that the formulas in (5.1) generalize to higher order tensors. We will shortly carry out the computation for (2, 0)–tensors in this way.

It is perhaps clear by now the reason for the name *covariant* derivative: Starting with a k–times covariant tensor ω, we end up with a $k+1$–times covariant tensor $\nabla\omega$, by "anti–contracting" with a vector field – direction of differentiation.

Chapter 6

Covariant Derivatives: (2, 0)–Tensors

Let now g be a (2, 0)–tensor, and let $g_{\lambda\mu} = g(\partial_\lambda, \partial_\mu)$ be its components, so that $g = g_{\lambda\mu}(\theta^\lambda \otimes \theta^\mu)$. By Leibnitz rule, we then have

$$D_\alpha g = D_\alpha(g_{\lambda\mu}\theta^\lambda) \otimes \theta^\mu + g_{\lambda\mu}\theta^\lambda \otimes D_\alpha(\theta^\mu).$$

Proceeding as before, we find that

$$\begin{aligned} D_\alpha[\theta^\mu(\partial_\gamma)] &= (D_\alpha\theta^\mu)(\partial_\gamma) + \theta^\mu(D_\alpha\partial_\gamma) \\ &= [D_\alpha\theta^\mu](\partial_\gamma) + \Gamma^\mu_{\alpha\gamma}. \end{aligned}$$

Since $\theta^\mu(\partial_\gamma) = \delta^\mu_\gamma$, the left hand side is equal to zero, and hence

$$[D_\alpha\theta^\mu](\partial_\gamma) = -\Gamma^\mu_{\alpha\gamma}.$$

Similarly,

$$\begin{aligned} D_\alpha[g_{\lambda\mu}\theta^\lambda(\partial_\beta)] &= D_\alpha(g_{\lambda\mu}\theta^\lambda)(\partial_\beta) + g_{\lambda\mu}\theta^\lambda[D_\alpha\partial_\beta] \\ &= D_\alpha(g_{\lambda\mu}\theta^\lambda)(\partial_\beta) + g_{\lambda\mu}\Gamma^\lambda_{\alpha\beta}. \end{aligned}$$

Since $D_\alpha[g_{\lambda\mu}\theta^\lambda(\partial_\beta)] = D_\alpha(g_{\lambda\mu}\delta^\lambda_\beta) = \partial_\alpha g_{\beta\mu}$, we learn that

$$D_\alpha(g_{\lambda\mu}\theta^\lambda)(\partial_\beta) = \partial_\alpha g_{\beta\mu} - g_{\lambda\mu}\Gamma^\lambda_{\alpha\beta}.$$

Thus, we learn that

$$D_\alpha g = (\partial_\alpha g_{\beta\mu} - g_{\lambda\mu}\Gamma^\lambda_{\alpha\beta})\theta^\beta \otimes \theta^\mu - g_{\lambda\mu}\Gamma^\mu_{\alpha\gamma}\theta^\lambda \otimes \theta^\gamma.$$

Renaming indices $\mu \rightsquigarrow \gamma, \lambda \rightsquigarrow \beta$, we see that

$$D_\alpha g = [\partial_\alpha g_{\beta\gamma} - g_{\lambda\gamma}\Gamma^\lambda_{\alpha\beta} - g_{\beta\mu}\Gamma^\mu_{\alpha\gamma}](\theta^\beta \otimes \theta^\gamma).$$

We note that this is a generalization of the second formula in (5.1).

Remark. The coefficient in the last display is the $(\alpha, \beta, \gamma)^{\text{th}}$ component of the (3, 0)–tensor ∇g, where if X, Y, Z are vector fields, then $\nabla g(X, Y, Z) = (D_X g)(Y, Z)$. By the usual abuse of notation (or poetic license), we may denote such a component by a host of different things:

$$(D_\alpha g)_{\beta\gamma} = (\nabla g)_{\alpha\beta\gamma} = \nabla_\alpha g_{\beta\gamma} = g_{\beta\gamma;\alpha}.$$

Chapter 7

Christoffel Symbols

Recall that the Christoffel symbols are defined by

$$D_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta} \partial \gamma.$$

Equivalently, $\Gamma^\gamma_{\alpha\beta} = (D_\alpha \partial_\beta)^\gamma = \theta^\gamma(D_\alpha \partial_\beta)$.

Let us consider two different local charts, or (from the "pointwise" perspective) a change of bases: $\{\partial_\alpha\}$ and $\{\tilde{\partial}_k\}$ such that

$$\partial_\alpha = A^k_\alpha \tilde{\partial}_k.$$

(Of course, from the "coordinate" perspective, it is clear that $A^k_\alpha = \frac{\partial x^k}{\partial x^\alpha}$.) So if X is a vector field, then we can write

$$X^\alpha A^k_\alpha \tilde{\partial}_k = X^\alpha \partial_\alpha = X = X^k \tilde{\partial}_k,$$

and if ω is a 1–form, then

$$\omega_\lambda B^\lambda_k \tilde{\theta}^k = \omega_\lambda \theta^\lambda = \omega = \omega_k \tilde{\theta}^k,$$

where (B^λ_k) is the matrix inverse of (A^k_λ).

Now let $\omega = \omega_\lambda \theta^\lambda = \omega_k \tilde{\theta}^k$ be a 1–form and let us express the scalar quantity $\omega[D_X(Y)]$ in both coordinate systems:

$$\begin{aligned}
\omega[D_X(Y)] &= \omega_\lambda \theta^\lambda X^\alpha D_\alpha(Y^\beta \partial_\beta) \\
&= \omega_\lambda X^\alpha [(\partial_\alpha Y^\lambda) + Y^\beta \Gamma^\lambda_{\alpha\beta}] \\
&= \omega_k X^i [(\partial_i Y^k) + Y^j \Gamma^k_{ij}].
\end{aligned}$$

On the other hand, changing basis explicitly, we also have

$$\begin{aligned}
\omega[D_X(Y)] &= \omega_\lambda B^\lambda_k X^\alpha A^i_\alpha [\partial_i (A^k_\gamma Y^\gamma) + Y^\beta A^j_\beta \Gamma^k_{ij}] \\
&= \omega_\lambda B^\lambda_k X^\alpha A^i_\alpha [Y^\gamma \partial_i A^k_\gamma + A^k_\gamma \partial_i Y^\gamma + Y^\beta A^j_\beta \Gamma^k_{ij}]
\end{aligned}$$

Choosing X and ω so that (for any particular indices λ and α), we have $\omega_\ell = \delta_{\lambda,\ell}, X^\ell = \delta^{\alpha,\ell}$, we get that

$$\partial_\alpha Y^\lambda + Y^\beta \Gamma^\lambda_{\alpha\beta} = B^\lambda_k A^i_\alpha [Y^\gamma \partial_i A^k_\gamma + A^k_\gamma \partial_i Y^\gamma + Y^\beta A^j_\beta \Gamma^k_{ij}]. \tag{7.1}$$

We can also rewrite

$$\partial_\alpha Y^\lambda = A^i_\alpha \partial_i Y^\lambda$$

and note that

$$B_k^\lambda A_\gamma^k = \delta_\gamma^\lambda,$$

and therefore in (7.1), $\partial_\alpha Y^\lambda$ cancels the corresponding term on the right hand side. Finally, choosing Y so that (for any particular index β), we have $Y^\ell = \delta^{\beta,\ell}$, we see that

$$\begin{aligned}\Gamma_{\alpha\beta}^\lambda &= B_k^\lambda A_\alpha^i[\partial_i A_\beta^k + A_\beta^j \Gamma_{ij}^k] \\ &= B_k^\lambda A_\alpha^i A_i^\gamma \partial_\gamma A_\beta^k + B_k^\lambda A_\alpha^i A_\beta^j \Gamma_{ij}^k \\ &= B_k^\lambda(\partial_\alpha A_\beta^k) + B_k^\lambda A_\alpha^i A_\beta^j \Gamma_{ij}^k,\end{aligned}$$

where the last equality follows directly since $A_\alpha^i A_i^\gamma = \delta_\alpha^\gamma$.

We see that $\Gamma_{\alpha\beta}^\lambda$ is not the component of a tensor, since that would mean the $B_k^\lambda(\partial_\alpha A_\beta^k)$ term should be absent. On the other hand, if $\tilde{\Gamma}_{\alpha\beta}^\lambda$ represents another connection, then by the same computation (since $B_k^\lambda(\partial_\alpha A_\beta^k)$ does not depend on the connection, only the coordinates)

$$C_{\alpha\beta}^\lambda = \tilde{\Gamma}_{\alpha\beta}^\lambda - \Gamma_{\alpha\beta}^\lambda$$

is indeed the component of a (2, 1)–tensor.

Conversely, we can add any (2, 1)–tensor to a connection to form another connection (since tensoriality ensures the relevant properties of a connection). Thus connections are unique up to adding (2, 1)–tensors.

Chapter 8

Riemannian Connection

A (pseudo) Riemannian manifold is a manifold equipped with a symmetric, invertible 2–form g. Here symmetric means $g(X,Y) = g(Y,X)$ and invertible means the matrix with entries $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$ is invertible.

A Riemannian connection is a connection which is torsion free and commutes with the covariant derivative, i.e., it satisfies

- $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$
- $\nabla g = 0$

Given these conditions, we can explicitly solve for the $\Gamma^\gamma_{\alpha\beta}$'s. From before, we have (cyclically permuting indices)

$$\begin{aligned}
0 = (D_\alpha g)_{\beta\gamma} &= \partial_\alpha g_{\beta\gamma} - g_{\lambda\gamma}\Gamma^\lambda_{\alpha\beta} - g_{\beta\lambda}\Gamma^\lambda_{\alpha\gamma} \\
0 = (D_\beta g)_{\gamma\alpha} &= \partial_\beta g_{\gamma\alpha} - g_{\lambda\alpha}\Gamma^\lambda_{\beta\gamma} - g_{\gamma\lambda}\Gamma^\lambda_{\beta\alpha} \\
0 = (D_\gamma g)_{\alpha\beta} &= \partial_\gamma g_{\alpha\beta} - g_{\lambda\beta}\Gamma^\lambda_{\gamma\alpha} - g_{\alpha\lambda}\Gamma^\lambda_{\gamma\beta}
\end{aligned} \tag{8.1}$$

Note that *a priori* there are $3! = 6$ such equations, but since $g_{\alpha\beta} = g_{\beta\alpha}$ and $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, we have also that

$$(D_\alpha g)_{\beta\gamma} = (D_\alpha g)_{\gamma\beta}$$

and hence we only have three equations. Solving these equations, we learn that

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}[\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}]g^{\lambda\gamma}. \tag{8.2}$$

We notice that equations (8.1) directly imply that the connection is *metric* (and vice versa), i.e.,

$$\partial_Z g(X,Y) = g(D_Z X, Y) + g(X, D_Z Y). \tag{8.3}$$

By tensoriality, it is enough to check this relation for coordinate vector fields:

$$\begin{aligned}
\partial_\gamma g(\partial_\alpha, \partial_\beta) &= \partial_\gamma g_{\alpha\beta} \\
&= g_{\lambda\beta}\Gamma^\lambda_{\gamma\alpha} + g_{\alpha\lambda}\Gamma^\lambda_{\gamma\beta} \\
&= g(\Gamma^\lambda_{\gamma\alpha}\partial_\lambda, \partial_\beta) + g(\partial_\alpha, \Gamma^\lambda_{\gamma\beta}\partial_\lambda) \\
&= g(D_\gamma\partial_\alpha, \partial_\beta) + g(\partial_\alpha, D_\gamma\partial_\beta).
\end{aligned}$$

Remark. Conversely, it is possible to first define the covariant derivative implicitly via the equation (which holds in $\mathbb{R}^d$)

$$2g(D_Y X, Z) = (L_X g)(Y, Z) + (d\theta_X)(Y, Z)$$

where L_X denotes the Lie derivative, and $\theta_X(Y) = g(X, Y)$, and then compute in coordinates to see that indeed the Christoffel symbols are as given in (8.2). This is done in [2].

Finally, we note that cyclically permuting indices in the last computation and adding two of the resulting equations and subtracting the third (and using that the connection is torsion free) we also learn that the Riemannian connection satisfies

$$\begin{aligned}\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} &= 2g(D_\alpha \partial_\beta, \partial_\gamma) \\ &= 2\Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma}.\end{aligned}$$

Finally, tautologically, we have

$$\begin{aligned}\Gamma^\lambda_{\alpha\beta} &= g(D_\alpha \partial_\beta, \partial_\gamma) g^{\lambda\gamma} \\ &:= \Gamma_{\alpha\beta,\gamma}\ g^{\lambda\gamma}.\end{aligned}$$

Chapter 9

Contraction Against g: Duality and Raising and Lowering of Indices

We note that contraction against $g_{\alpha\beta}$ dualizes the tensor and hence lowers indices (since indices are placed oppositely of the actual object), whereas contraction against $g^{\alpha\beta}$ has the opposite effect. That is, contraction against $g_{\alpha\beta}$ sends (p,q)–tensors to $(p+1,q-1)$–tensors, whereas contraction against $g^{\alpha\beta}$ sends (p,q)–tensors to $(p-1,q+1)$–tensors.

First we have that

$$X^\alpha g_{\alpha\beta} = g(X,\partial_\beta) := X_\beta$$

represents the 1–form dual to the vector field $X = X^\alpha\partial_\alpha$:

$$X_\beta\theta^\beta : Y \to g(Y,X),$$

i.e.,

$$X_\beta\theta^\beta(Y) = Y^T[g]X = Y^\beta g_{\alpha\beta}X^\alpha = Y^\beta X_\beta = g(X,Y).$$

In particular, $X^\beta g_{\alpha\beta}X^\alpha = g(X,X)$.

Thought another way, (in matrix form) $[g][X^\beta]$ gives the column vector $[g(\partial_\alpha,X)]$ (here $[X^\beta]$ denotes the column vector with entries being components of X) so $[g]^{-1}[g][X^\beta] = [X]$, i.e.,

$$g^{\gamma\alpha}g_{\alpha\beta}X^\beta = X^\gamma,$$

or

$$X = g^{\gamma\alpha}g(\partial_\alpha,X)\partial_\gamma$$

In the particular cases when $\{\partial_\alpha\}$ form an orthonormal basis with respect to g, we have $[g] = \mathrm{Id}$, so that

$$X = g(\partial_\alpha,X)\partial_\alpha.$$

The dual situation is if $\omega_\alpha\theta^\alpha$ is a 1–form, then $X^\beta = g^{\alpha\beta}\omega_\alpha$ are the components of a vector. What we have is nothing other than the usual duality provided by g: We have maps

$$T_pM \to T_p^*M : X \mapsto \omega(X) : \omega(X)(\cdot) = g(X,\cdot) \qquad \text{(lowering of indices)}$$

and its inverse

$$T_p^*M \to T_pM : \omega \mapsto X(\omega) : \omega(\cdot) = g(\cdot,X(\omega)) \qquad \text{(raising of indices)}$$

($X(\omega)$ is exactly the vector provided by the Riesz Representation Theorem.)

Indeed, we have that $\omega(\partial_\alpha) = \omega_\alpha$, and thus the vector $X(\omega)$ satisfies

$$g(\partial_\alpha, X(\omega)) = \omega_\alpha$$

and thus as observed before, we have

$$[g]^{-1}[\omega_\alpha] = [X(\omega)] : g^{\beta\alpha}\omega_\alpha = (X(\omega))^\beta.$$

Chapter 10

Laplacian (For Functions)

10.1 Contravariant Derivative and Gradient

By raising indices with respect to $g^{\alpha\beta}$ (dualizing) we obtain the contravariant derivative:

$$D^{\beta} = g^{\alpha\beta} D_{\alpha}.$$

Let us first understand this in the case of a function, which we view as a 0–form. In accord with the notation we have used to denote the covariant derivative, ∇f is a 1–form:

$$\nabla f = (\nabla f)_{\alpha} \theta^{\alpha} = (D_{\alpha} f) \theta^{\alpha} := (\nabla_{\alpha} f) \theta^{\alpha},$$

so that $\nabla_{\alpha} f$ denote components of a 1–form. Given a metric tensor g, the usual dualizing gives a vector field $X(\nabla f)$ such that for all α,

$$\nabla_{\alpha} f = g(X(\nabla f), \partial_{\alpha}).$$

Therefore,

$$X(\nabla f) = (g^{\alpha\gamma} \nabla_{\alpha} f) \partial_{\gamma} = (g^{\alpha\gamma} D_{\alpha} f) \partial_{\gamma} = (D^{\gamma} f) \partial_{\gamma} := (\nabla^{\gamma} f) \partial_{\gamma},$$

so that $\nabla^{\gamma} f$ denote the components of a vector field. Finally, by the usual abuse of notation, we will denote

$$X(\nabla f) = \nabla f = (\nabla^{\gamma} f) \partial_{\gamma}.$$

10.2 Laplacian as Trace of Hessian

We define the Laplacian

$$\triangle_g f := \operatorname{tr}(\nabla(\nabla f)) = \theta^{\alpha}(\nabla_{\alpha}(\nabla f))$$

i.e., it is the sum of the α component of the *covariant* derivatives of ∇f in the direction ∂_{α}.

A more general formulation is as follows: Let us consider the second order operator defined implicitly as

$$\begin{aligned} g(\nabla_X \nabla f, Y) &= D_X g(\nabla f, Y) - g(\nabla f, D_X Y) \\ &= D_X D_Y f - D_{D_X Y} f. \end{aligned}$$

(Here we have used the fact that covariant differentiation is commensurate with the metric g.) This indeed implicitly defines the Hessian of f

$$\text{Hess}f(X,Y) = g(\nabla_X \nabla f, Y)$$

and further, the definition

$$(\text{Hess}f(X,Y) =)\nabla^2_{X,Y} f = (D_X D_Y - D_{D_X Y})f$$

can be easily generalized to $(p,0)$–tensors as the *second covariant derivative*, which is then a $(p+2,0)$ tensor (here we are viewing f as a 0–form). It is easy to check that for *functions*, the second covariant derivative is symmetric. Indeed, let $h_{\alpha\beta}$ denote components of the Hessian, then

$$h_{\alpha\beta} = g(D_\alpha \nabla f, \partial_\beta) = \partial_\alpha \partial_\beta f - \Gamma^\gamma_{\alpha\beta}\partial_\gamma f,$$

and since the connection is torsion free and $\partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha$, we can clearly interchange α and β in the above. Finally, we can retrieve $\triangle_g f$ as the *trace* of Hess(f), i.e.,

$$\begin{aligned} h_{\alpha\beta} g^{\alpha\beta} &= g^{\alpha\beta} g(D_\alpha \nabla f, \partial_\beta) \\ &= g^{\alpha\beta}(D_\alpha \nabla f)^\gamma g_{\gamma\beta} \\ &= \delta^\alpha_\gamma (D_\alpha \nabla f)^\gamma \\ &= \theta^\alpha (D_\alpha \nabla f). \end{aligned}$$

(In the particular case where we have an orthonormal frame, we have $\triangle_g f = \sum_\alpha h_{\alpha\alpha}$.)

Let us now compute $\triangle_g f$ in coordinates. We have

$$\begin{aligned} \triangle_g f &= \theta^\alpha(\nabla_\alpha(\nabla f)) \\ &= \theta^\alpha[\partial_\alpha(g^{\beta\gamma}\partial_\beta f)\partial_\gamma + (g^{\beta\gamma}\partial_\beta f)\Gamma^\lambda_{\alpha\gamma}\partial_\lambda] \\ &= g^{\alpha\beta}\partial_\alpha\partial_\beta f + \partial_\alpha g^{\alpha\beta}\partial_\beta f + (g^{\beta\gamma}\partial_\beta f)\delta^\alpha_\lambda \Gamma^\lambda_{\alpha\gamma}. \end{aligned}$$

Now using the relation that

$$\Gamma^\lambda_{\alpha\gamma} = \frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\gamma\mu} + \partial_\gamma g_{\mu\alpha} - \partial_\mu g_{\alpha\gamma}),$$

the last term in the above display becomes

$$\begin{aligned} \frac{1}{2}(\partial_\beta f) g^{\beta\gamma}\delta^\alpha_\lambda g^{\mu\lambda}(\partial_\alpha g_{\gamma\mu} + \partial_\gamma g_{\mu\alpha} - \partial_\mu g_{\alpha\gamma}) &= \frac{1}{2}(\partial_\beta f) g^{\beta\gamma} g^{\mu\lambda}(\partial_\lambda g_{\gamma\mu} + \partial_\gamma g_{\mu\lambda} - \partial_\mu g_{\lambda\gamma}) \\ &= \frac{1}{2}(\partial_\beta f) g^{\beta\gamma} g^{\mu\lambda}(\partial_\gamma g_{\mu\lambda}), \end{aligned}$$

where to obtain the first equality we have performed the trace δ^α_λ and to obtain the second equality, we note that interchanging λ and μ and using that g is symmetric, we have

$$g^{\mu\lambda}\partial_\lambda g_{\gamma\mu} = g^{\lambda\mu}\partial_\mu g_{\gamma\lambda} = g^{\mu\lambda}\partial_\mu g_{\lambda\gamma}$$

and hence the only term that remains is $g^{\mu\lambda}(\partial_\gamma g_{\mu\lambda})$. We have thus obtained the expression (after $\gamma \rightsquigarrow \alpha$)

$$\triangle_g f = \left[g^{\alpha\beta}\partial_\alpha + \partial_\alpha g^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g^{\mu\lambda}(\partial_\alpha g_{\mu\lambda}) \right] \partial_\beta f$$

10.3 Laplacian as $div(\nabla f)$

On the other hand, we may start with the volume form (see e.g., [3])

$$dv = \sqrt{|g|}\ dx^1 \wedge \cdots \wedge dx^n$$

(Here $|g| = \det([g])$) which induces *global* inner products

$$\langle X, Y\rangle = \int_M g(X,Y)\ dv, \quad \langle f, g\rangle = \int_M fh\ dv.$$

Here X, Y are vector fields and f, h are functions. Now via integration by parts we may implicitly define divergence as

$$\langle -\mathrm{div}X, f\rangle = \langle X, \nabla f\rangle$$

which by explicit computation gives

$$\mathrm{div}X = \frac{1}{\sqrt{|g|}}\partial_\alpha(X^\alpha\sqrt{|g|}).$$

Indeed, with $X = X^\alpha\partial_\alpha$,

$$\begin{aligned}
\langle X, \nabla f\rangle &= \int g(X, \nabla f)\ dv \\
&= \int X^\alpha \partial_\alpha f\sqrt{|g|}\ dx^1 \wedge \cdots \wedge dx^n \\
&= -\int \partial_\alpha(X^\alpha\sqrt{|g|})f\ dx^1 \wedge \cdots \wedge dx^n \\
&= -\int \frac{1}{\sqrt{|g|}}\partial_\alpha(X^\alpha\sqrt{|g|})f\ dv \\
&= -\int (\mathrm{div}X)\cdot f\ dv
\end{aligned}$$

Plugging in $X^\alpha = \nabla^\alpha f = g^{\alpha\beta}\partial_\beta f$, we obtain that

$$\triangle_g f = \frac{1}{\sqrt{|g|}}\partial_\alpha(g^{\alpha\beta}\sqrt{|g|}\partial_\beta f)$$

Let us check that this agrees with our previous definition. We have

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\partial_\alpha(g^{\alpha\beta}\sqrt{|g|}\partial_\beta f) &= \left[g^{\alpha\beta}\partial_\alpha + \partial_\alpha g^{\alpha\beta} + \frac{\partial_\alpha\sqrt{|g|}}{\sqrt{|g|}}g^{\alpha\beta}\right]\partial_\beta f \\
&= \left[g^{\alpha\beta}\partial_\alpha + \partial_\alpha g^{\alpha\beta} + \frac{1}{2}\frac{\partial_\alpha|g|}{|g|}g^{\alpha\beta}\right]\partial_\beta f.
\end{aligned}$$

Now we recall Jacobi's Formula

$$\partial_{g_{\mu\lambda}}|g| = |g|g^{\mu\lambda}.$$

Thus we have

$$\frac{\partial_\alpha|g|}{|g|} = \frac{\partial_{g_{\mu\lambda}}\partial_\alpha g_{\mu\lambda}}{|g|} = g^{\mu\lambda}\partial_\alpha g_{\mu\lambda}$$

and the two definitions of the Laplacian are the same.

10.4 Other (Coordinate) Expressions

Recall the expression

$$\triangle_g f = \left[g^{\alpha\beta}\partial_\alpha + \partial_\alpha g^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g^{\mu\lambda}(\partial_\alpha g_{\mu\lambda}) \right] \partial_\beta f.$$

We will now rewrite this expression in terms of the Christoffel symbols. We note that

$$g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma = \frac{1}{2} g^{\alpha\beta} g^{\gamma\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta})\partial_\gamma.$$

Now we use the fact that

$$\partial_\mu(g^{\alpha\beta} g_{\beta\gamma}) = \partial_\mu \delta^\alpha_\gamma = 0$$

to rewrite the first two terms in the previous expression as

$$\begin{aligned}
\Gamma^\gamma\partial_\gamma := g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma &= -\frac{1}{2} g^{\alpha\beta}(g_{\beta\lambda}\partial_\alpha g^{\gamma\lambda} + g_{\lambda\alpha}\partial_\beta g^{\gamma\lambda} + g^{\gamma\lambda}\partial_\lambda g_{\alpha\beta})\partial_\gamma \\
&= -\frac{1}{2}(\delta^\alpha_\lambda \partial_\alpha g^{\gamma\lambda} + \delta^\beta_\lambda \partial_\beta g^{\gamma\lambda} + g^{\alpha\beta} g^{\gamma\lambda}\partial_\lambda g_{\alpha\beta})\partial_\gamma \\
&= -(\partial_\alpha g^{\alpha\gamma} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\lambda}\partial_\lambda g_{\alpha\beta})\partial_\gamma \\
&= -(\partial_\alpha g^{\alpha\gamma} + \frac{1}{2} g^{\gamma\lambda} g^{\alpha\beta}\partial_\lambda g_{\alpha\beta})\partial_\gamma.
\end{aligned}$$

(Incidentally, by symmetry of g,

$$\frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\mu\lambda}) = \frac{1}{2} g^{\mu\lambda}(\Gamma^\gamma_{\alpha\mu} g_{\lambda\gamma} + \Gamma^\gamma_{\alpha\lambda} g_{\mu\gamma}) = g^{\mu\lambda}\Gamma^\gamma_{\alpha\mu} g_{\lambda\gamma} = \Gamma^\mu_{\alpha\mu}.)$$

Denoting by

$$\widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$$

the *reduced wave operator*, we see that we have

$$\triangle_g = \widetilde{\Box}_g - \Gamma^\gamma\partial_\gamma.$$

Alternatively,

$$\triangle_g f = D^\alpha(\partial_\alpha f) - (D^\alpha\partial_\alpha)(f).$$

Chapter 11

Covariant Derivative for General Tensors

Let us recall that the covariant derivative preserves the type of tensor, and starting with the connection $D_X Y$ we are able to find the covariant derivative of 1–forms since D_X commutes with evaluation: If ω is a 1–form, then

$$D_X[(\omega(Y)] = (D_X\omega)(Y) + \omega(D_X(Y))$$

Given that D_X also satisfies Leibnitz rule:

$$D_X(u \otimes v) = (D_X u) \otimes v + u \otimes (D_X v),$$

we should be able to compute the covariant derivative of general (p, q)–tensors by induction. We will do so here and along the way develop some notation to handle higher order tensors.

First let us recall that if $X = X^\alpha \partial_\alpha$ is a vector field and $\omega = \omega_\alpha \theta^\alpha$ is a 1–form, then

$$\begin{aligned} D_\beta X &= (\partial_\beta X^\lambda + X^\alpha \Gamma^\lambda_{\beta\alpha})\partial_\lambda \\ D_\beta \omega &= (\partial_\beta \omega_\lambda - \omega_\gamma \Gamma^\gamma_{\beta\lambda})\theta^\lambda. \end{aligned}$$

We will shortly see that with appropriate notation the above form easily generalize to $(p, 0)$ and $(0, q)$–tensors, respectively. To start we will use multi–index:

$$[\alpha] := \alpha_1, \alpha_2, \ldots, \alpha_p$$

and write e.g.,

$$X^{[\alpha]}\partial_{[\alpha]} := X^{\alpha_1,\alpha_2,\ldots,\alpha_p}(\partial_{\alpha_1} \otimes \partial_{\alpha_2} \otimes \cdots \otimes \partial_{\alpha_p}).$$

First we let T be a $(0, q)$–tensor so that

$$T = T^{[\alpha]}\partial_{[\alpha]}.$$

Applying the Leibnitz rule once, we get that

$$D_\beta T = (\partial_\beta T^{[\alpha]})\partial_{[\alpha]} + T^{[\alpha]}(D_\beta \partial_{[\alpha]}).$$

To understand the second term, let us for now suppose $p = 2$, so that we have

$$\begin{aligned} D_\beta(\partial_{\alpha_1} \otimes \partial_{\alpha_2}) &= D_\beta \partial_{\alpha_1} \otimes \partial_{\alpha_2} + \partial_{\alpha_1} \otimes D_\beta \partial_{\alpha_2} \\ &= \Gamma^\gamma_{\beta\alpha_1}\partial_\gamma \otimes \partial_{\alpha_2} + \partial_{\alpha_1} \otimes \Gamma^\gamma_{\beta\alpha_2}\partial_\gamma. \end{aligned}$$

This pattern clearly persists if $p > 2$: Namely, we will have sum of terms where $\Gamma^{\gamma}_{\beta\alpha_k}\partial_\gamma$ will replace the γ^{th} spot in the tensor product. Let us now introduce the notation

$$\partial^{(\alpha_k)}_{[\alpha]'} \circ (\Gamma^{\gamma}_{\beta\alpha_k}\partial_\gamma)$$

to mean exactly this. The parenthesis around α_k in the superscript denotes that the term following the $\circ$ should be inserted in the k^{th} spot, and the α_k's are summed as usual (ranging over all single indices contained in $[\alpha]$). Thus we learn that

$$D_\beta(T^{[\alpha]}\partial_{[\alpha]}) = (\partial_\beta T^{[\alpha]})\partial_{[\alpha]} + T^{[\alpha]}\partial^{(\alpha_k)}_{[\alpha]} \circ (\Gamma^{\gamma}_{\beta\alpha_k}\partial_\gamma).$$

We would now like to to re–sum so that both terms isolate the index γ. For this purpose let us introduce the assignment operator:

$$T^{[\alpha]'k:\gamma} := T^{[\alpha]}\delta^{\alpha_k,\gamma}$$

so that (using multi–linearity)

$$D_\beta(T^{[\alpha]}\partial_{[\alpha]}) = \left[\partial_\beta T^{[\alpha]'k:\gamma} + T^{[\alpha]}\Gamma^{\gamma}_{\beta\alpha_k}\right](\partial^{(\alpha_k)}_{[\alpha]} \circ \partial_\gamma).$$

Next let S be a $(p, 0)$–tensor so that

$$S = S_{[\alpha]}\theta^{[\alpha]}.$$

Proceeding as before we have

$$\begin{aligned} D_\beta S &= (\partial_\beta S_{[\alpha]})\theta^{[\alpha]} + S_{[\alpha]}(D_\beta \theta^{[\alpha]}) \\ &= (\partial_\beta S_{[\alpha]})\theta^{[\alpha]} + S_{[\alpha]}\theta^{[\alpha]}_{(\alpha_k)} \circ (-\Gamma^{\alpha_k}_{\beta\gamma}\,\theta^\gamma) \\ &= \left[(\partial_\beta S_{[\alpha]'k:\gamma}) - S_{[\alpha]}\Gamma^{\alpha_k}_{\beta\gamma}\right](\theta^{[\alpha]}_{(\alpha_k)} \circ \theta^\gamma). \end{aligned}$$

Finally, if R is now a general (p, q)–tensor so that

$$R = R^{[\mu]}_{[\alpha]}(\theta^{[\alpha]} \otimes \partial_{[\mu]}),$$

then we have

$$\begin{aligned} D_\beta R &= (\partial_\beta R^{[\mu]}_{[\alpha]})(\theta^{[\alpha]} \otimes \partial_{[\mu]}) + R^{[\mu]}_{[\alpha]} D_\beta(\theta^{[\alpha]} \otimes \partial_{[\mu]}) \\ &= (\partial_\beta R^{[\mu]}_{[\alpha]})(\theta^{[\alpha]} \otimes \partial_{[\mu]}) + R^{[\mu]}_{[\alpha]}\left[\theta^{[\alpha]}_{(\alpha_k)} \circ (-\Gamma^{\alpha_k}_{\beta\gamma}\theta^\gamma) \otimes \partial_{[\mu]} + \theta^{[\alpha]} \otimes D_\beta \partial_{[\mu]}\right] \\ &= (\partial_\beta R^{[\mu]}_{[\alpha]})(\theta^{[\alpha]} \otimes \partial_{[\mu]}) + R^{[\mu]}_{[\alpha]}\left[\theta^{[\alpha]}_{(\alpha_k)} \circ (-\Gamma^{\alpha_k}_{\beta\gamma}\theta^\gamma) \otimes \partial_{[\mu]} + \theta^{[\alpha]} \otimes \partial^{(\mu_\ell)}_{[\mu]} \circ \Gamma^{\gamma}_{\beta\mu_\ell}\partial_\gamma\right] \\ &= (\partial_\beta R^{[\mu]}_{[\alpha]})(\theta^{[\alpha]} \otimes \partial_{[\mu]}) + R^{[\mu]}_{[\alpha]}\left[\Gamma^{\lambda}_{\beta\mu_\ell}\theta^{[\alpha]} \otimes (\partial^{(\mu_\ell)}_{[\mu]} \circ \partial_\lambda) - \Gamma^{\alpha_k}_{\beta\gamma}(\theta^{[\alpha]}_{(\alpha_k)} \circ \theta^\gamma) \otimes \partial_{[\mu]}\right] \\ &= \left[\partial_\beta R^{[\mu]'k:\gamma}_{[\alpha]'\ell:\lambda} + R^{[\mu]}_{[\alpha]'k:\gamma}\,\Gamma^{\lambda}_{\beta\alpha_\ell} - R^{[\mu]'\ell:\lambda}_{[\alpha]}\,\Gamma^{\alpha_k}_{\beta\gamma}\right][(\theta^{[\alpha]}_{(\alpha_k)}, \partial^{(\alpha_\ell)}_{[\mu]}) \circ (\theta^\gamma, \partial_\lambda)] \end{aligned}$$

Chapter 12

Change of Variables, Pullback Metric, and Normal Coordinates

In many cases it will be convenient to use the *normal coordinate system* (by tensoriality, it is sufficient to prove a tensor identity in any convenient coordinate system). Such a coordinate system at a point P is a local coordinate system such that

$$g_{\alpha\beta}(P) = \delta_\alpha^\beta \quad \text{and} \quad \partial_\gamma g_{\alpha\beta}(P) = 0, \text{for all } \alpha, \beta, \gamma,$$

that is, we require orthonormality and vanishing of the Christoffel symbols at the point P.

It is straightforward to construct such a coordinate system:

∘ Choose a local chart φ around P so that $\varphi(P) = 0$ and let $\{x^\alpha\}$ denote the local coordinate system.

∘ Via a linear transformation, choose a frame in $\mathbf{R}^n$ so that $g_{\alpha\beta}(P) = \delta_\alpha^\beta$. Here g is the appropriate *pullback* (see below) of the Euclidean inner product.

∘ Change coordinates via a map $F : \{x^\alpha\} \to \{y^\beta\}$ by

$$x^\lambda - y^\lambda = -\Gamma^\lambda_{\alpha\beta}(P) y^\alpha y^\beta$$

and note that at P, $[DF] = \text{Id}$, where $[DF]$ denotes the Jacobian matrix, so that by the inverse function theorem $\{y^\beta\}$ are coordinates.

∘ We will now need to consider the *pullback* metric, which in a general context is defined as follows: If $F : (M, g_M, x) \to (N, g_N, y)$, then

$$g_M(X, Y) = g_N(F_*(X), F_*(Y)).$$

Here $F_* : T_pM \to T_{F(p)}M$ is the *pushforward* of F, defined so that if $h : N \to \mathbf{R}$, then

$$[X(h \circ F)](p) = [F_*(X)(h)](F(p)).$$

By the chain rule, we learn that

$$\frac{\partial(h \circ F)}{\partial x^\alpha}(p) = \frac{\partial h}{\partial y^\beta} \frac{\partial y^\beta}{\partial x^\alpha}(F(p)).$$

That is,

$$\partial_\alpha = (DF)^\beta_\alpha \tilde{\partial}_\beta$$

or inverting,

$$\tilde{\partial}_\beta = ((DF)^{-1})^\alpha_\beta \partial_\alpha = (DF^{-1})^\alpha_\beta \partial_\alpha,$$

where the last equality follows by the inverse function theorem. Note that DF has one covariant and one contravaiant index: The vector fields transform covariantly (in the same direction as the function F) whereas the coordinates themselves transform contravariantly. Notice also that strictly speaking we have "identified" $[DF]$ and $[DF]^T$, since it is clear that e.g., $(DF)^\beta_\alpha$ means $\frac{\partial y^\beta}{\partial x_\alpha}$.

Thus if we change variables from x to y and $F : (M, \tilde{g}, y) \to (N, g, x)$, then in the new coordinates the components of the metric tensor are given by

$$\tilde{g}_{\alpha\beta} = g((DF^{-1})^\lambda_\alpha \partial_\lambda, (DF^{-1})^\mu_\beta \partial_\mu).$$

∘ In our case, we have

$$[F(y)]^\lambda = x^\lambda = y^\lambda - \Gamma^\lambda_{\alpha\beta} y^\alpha y^\beta,$$

so

$$(DF^{-1})^\lambda_\mu = \delta^\lambda_\mu - \Gamma^\lambda_{\alpha\mu} y^\alpha$$

and (recall that $\tilde{g}$ is the metric in the new coordinates $\{y^\beta\}$).

$$\begin{aligned}
\tilde{g}(\tilde{\partial}_\alpha, \tilde{\partial}_\beta) &= g((DF^{-1})^\mu_\alpha \partial_\mu, (DF^{-1})^\lambda_\beta \partial_\lambda) \\
&= (\delta^\mu_\alpha - \Gamma^\mu_{\gamma\alpha} y^\gamma)(\delta^\lambda_\beta - \Gamma^\lambda_{\gamma\beta} y^\gamma) g_{\mu\lambda} \\
&= [\delta^\mu_\alpha \delta^\lambda_\beta - \delta^\mu_\alpha \Gamma^\lambda_{\gamma\beta} y^\gamma - \delta^\lambda_\beta \Gamma^\mu_{\gamma\alpha} y^\gamma + \Gamma^\mu_{\gamma\alpha}\Gamma^\lambda_{\gamma\beta}(y^\gamma)^2] g_{\mu\lambda} \\
&= g_{\alpha\beta} - (g_{\alpha\lambda}\Gamma^\lambda_{\gamma\beta} + g_{\mu\beta}\Gamma^\mu_{\gamma\alpha}) y^\gamma + \Gamma^\mu_{\gamma\alpha}\Gamma^\lambda_{\gamma\beta}(y^\gamma)^2 g_{\mu\lambda}.
\end{aligned}$$

Now we recall that at the point P, $DF = \text{Id}$ and so $y = x = \varphi(P) = 0$ and therefore

$$\begin{aligned}
\partial_\gamma \tilde{g}_{\alpha\beta} &= \partial_\gamma g_{\alpha\beta} - (g_{\alpha\lambda}\Gamma^\lambda_{\gamma\beta} + g_{\mu\beta}\Gamma^\mu_{\gamma\alpha}) \\
&= 0,
\end{aligned}$$

where again we have used the fact that $\partial_\gamma g(\partial_\alpha, \partial_\beta) = g(D_\gamma \partial_\alpha, \partial_\beta) + g(\partial_\alpha, D_\gamma \partial_\beta)$.

Chapter 13

(Riemannian) Curvature Tensors and Bianchi Identities

Next let us consider the curvature

$$R(X,Y)Z = (D_X D_Y - D_Y D_X - D_{[X,Y]})Z$$

and the associated $(3,1)$–tensor $R^\lambda_{\gamma\alpha\beta}(\theta^\gamma \otimes \theta^\alpha \otimes \theta^\beta \otimes \partial_\lambda)$ where

$$R^\lambda_{\gamma\alpha\beta} = [R(\partial_\alpha, \partial_\beta)\partial_\gamma]^\lambda = \partial_\alpha \Gamma^\lambda_{\beta\gamma} - \partial_\beta \Gamma^\lambda_{\alpha\gamma} + \Gamma^\mu_{\beta\gamma}\Gamma^\lambda_{\alpha\mu} - \Gamma^\mu_{\alpha\gamma}\Gamma^\lambda_{\beta\mu}.$$

In a normal coordinate system, the last two terms vanish, and we get that

$$R^\lambda_{\gamma\alpha\beta} = \partial_\alpha \Gamma^\lambda_{\beta\gamma} - \partial_\beta \Gamma^\lambda_{\alpha\gamma}.$$

Cyclically permuting α, β, γ, we also get

$$R^\lambda_{\alpha\beta\gamma} = \partial_\beta \Gamma^\lambda_{\gamma\alpha} - \partial_\gamma \Gamma^\lambda_{\beta\alpha}$$
$$R^\lambda_{\beta\gamma\alpha} = \partial_\gamma \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\gamma\beta}.$$

Adding these three equations and using the fact that the connection is torsion free, we get the *first Bianchi identity*

$$R^\lambda_{\gamma\alpha\beta} + R^\lambda_{\alpha\beta\gamma} + R^\lambda_{\beta\gamma\alpha} = 0.$$

Next we will differentiate the curvature, to obtain

$$\partial_\mu R^\lambda_{\gamma\alpha\beta} = \partial_{\mu\alpha} \Gamma^\lambda_{\beta\gamma} - \partial_{\mu\beta} \Gamma^\lambda_{\alpha\gamma}.$$

Cyclically permuting μ, α, β, we also get

$$\partial_\alpha R^\lambda_{\gamma\beta\mu} = \partial_{\alpha\beta} \Gamma^\lambda_{\mu\gamma} - \partial_{\alpha\mu} \Gamma^\lambda_{\beta\gamma}$$
$$\partial_\beta R^\lambda_{\gamma\mu\alpha} = \partial_{\beta\mu} \Gamma^\lambda_{\alpha\gamma} - \partial_{\beta\alpha} \Gamma^\lambda_{\mu\gamma}.$$

Adding these three equations and using the fact that $\partial_{\alpha\beta} = \partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$, we get the *second Biachi identity*

$$\partial_\mu R^\lambda_{\gamma\alpha\beta} + \partial_\alpha R^\lambda_{\gamma\beta\mu} + \partial_\beta R^\lambda_{\gamma\mu\alpha} = 0.$$

If we lower index, then we get

$$g_{\mu\lambda} R^\lambda_{\gamma\alpha\beta} = R_{\mu\gamma\alpha\beta} = g(R(\partial_\alpha, \partial_\beta)\partial_\gamma, \partial_\mu),$$

which are components of a (4, 0) Riemannian curvature tensor

$$X, Y, Z, W \mapsto g(R(X, Y)Z, W) := R(X, Y, Z, W).$$

Note that from the explicit expression for the components, we see that

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z).$$

Since the upper index in $R^{\lambda}_{\gamma\alpha\beta}$ is unchanged in derivation of the Bianchi identities, we recover the Bianchi identities for these dualized tensors

$$\begin{aligned} R_{\mu\gamma\alpha\beta} + R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} &= 0 \\ \partial_\sigma R_{\mu\gamma\alpha\beta} + \partial_\alpha R_{\mu\gamma\beta\sigma} + \partial_\beta R_{\mu\gamma\sigma\alpha} &= 0. \end{aligned}$$

Using the first Bianchi identity together with the symmetry property derived, we also obtain that R is symmetric in the first two and last two entries

$$R(X, Y, Z, W) = R(Z, W, X, Y).$$

Taking trace of the α and μ components in $R_{\mu\gamma\alpha\beta}$, we obtain (components of) the *Ricci tensor*:

$$\begin{aligned} R_{\beta\gamma} &= g^{\alpha\mu} R_{\mu\gamma\alpha\beta} \\ &= g^{\alpha\mu} g(R(\partial_\alpha, \partial_\beta)\partial_\gamma, \partial_\mu) \\ &= [R(\partial_\alpha, \partial_\beta)\partial_\gamma)]^\alpha. \end{aligned}$$

Since $R_{\mu\gamma\alpha\beta} = R_{\alpha\beta\mu\gamma}$, we easily see that

$$R_{\beta\gamma} = R_{\gamma\beta},$$

i.e., the Ricci tensor is symmetric. Also, we have the explicit expression

$$\begin{aligned} R_{\gamma\beta} &= g^{\mu\alpha} R_{\mu\gamma\alpha\beta} \\ &= g^{\mu\alpha} g_{\mu\lambda}(\partial_\alpha \Gamma^{\lambda}_{\beta\gamma} - \partial_\beta \Gamma^{\lambda}_{\gamma\alpha}) \\ &= \delta^{\alpha}_{\lambda}(\partial_\alpha \Gamma^{\lambda}_{\beta\gamma} - \partial_\beta \Gamma^{\lambda}_{\gamma\alpha}) \\ &= \partial_\alpha \Gamma^{\alpha}_{\beta\gamma} - \partial_\beta \Gamma^{\alpha}_{\gamma\alpha}. \end{aligned}$$

Finally, taking the trace one last time, we obtain the *scalar curvature* $R = g^{\beta\gamma} R_{\beta\gamma}$.

Chapter 14

Harmonic/Wave Coordinate Condition and Consequences

By (locally) solving the Dirichlet problem, we may choose harmonic coordinates x^λ so that $\triangle_g x^\lambda = 0$. In these coordinates we see that

$$\begin{aligned}\triangle_g x^\lambda &= [g^{\alpha\beta}\partial_\alpha\partial_\beta + \Gamma^\gamma\partial_\gamma]x^\lambda \\ &= \Gamma^\lambda \\ &= 0.\end{aligned}$$

Therefore in harmonic coordinates we see that

$$\triangle_g = \Box_g = \widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta.$$

Next we can see that the Ricci tensor also simplifies, up to $O((\partial g)^2)$. We have

$$\begin{aligned}R_{\gamma\beta} &= \partial_\alpha\Gamma^\alpha_{\beta\gamma} - \partial_\beta\Gamma^\alpha_{\gamma\alpha} + O((\partial g)^2) \\ &= \frac{1}{2}g^{\alpha\lambda}[\partial_\alpha(\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) - \partial_\beta(\partial_\alpha g_{\gamma\lambda} + \partial_\gamma g_{\lambda\alpha} - \partial_\lambda g_{\alpha\gamma})] + O((\partial g)^2) \\ &= \frac{1}{2}g^{\alpha\lambda}[\partial_\alpha(\partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) - \partial_\beta(\partial_\gamma g_{\lambda\alpha} - \partial_\lambda g_{\alpha\gamma})] + O((\partial g)^2) \\ &= -\frac{1}{2}g^{\alpha\lambda}\partial_\alpha\partial_\lambda g_{\beta\gamma} + \frac{1}{2}g^{\alpha\lambda}(\partial_\alpha\partial_\gamma g_{\lambda\beta} - \partial_\beta\partial_\gamma g_{\lambda\alpha} + \partial_\beta\partial_\lambda g_{\alpha\gamma}) + O((\partial g)^2)\end{aligned}$$

(Note that we are no longer in a normal coordinate system, but the $O(\Gamma^2)$ terms are also $O((\partial g)^2)$.) Now let us observe that given the wave coordinate condition that $g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0$, we have

$$\begin{aligned}g^{\alpha\lambda}\partial_\gamma(\partial_\alpha g_{\lambda\beta} - \partial_\beta g_{\lambda\alpha} + \partial_\lambda g_{\alpha\beta}) &= g^{\alpha\lambda}\partial_\gamma(\Gamma^\mu_{\alpha\lambda}g_{\beta\mu}) \\ &= g^{\alpha\lambda}g_{\beta\mu}\partial_\gamma\Gamma^\mu_{\alpha\lambda} \\ &= O((\partial g)^2)\end{aligned}$$

Thus we may continue the expression for $R_{\beta\gamma}$ as

$$\begin{aligned}R_{\beta\gamma} &= -\frac{1}{2}g^{\alpha\lambda}\partial_\alpha\partial_\lambda g_{\beta\gamma} + \frac{1}{2}g^{\alpha\lambda}\partial_\gamma(\partial_\alpha g_{\lambda\beta} - \partial_\beta g_{\lambda\alpha} + \partial_\lambda g_{\alpha\beta}) + \frac{1}{2}g^{\alpha\lambda}(\partial_\beta\partial_\lambda g_{\alpha\gamma} - \partial_\gamma\partial_\lambda g_{\alpha\beta}) + O((\partial g)^2) \\ &= -\frac{1}{2}\widetilde{\Box}_g\, g_{\beta\gamma} + \frac{1}{2}g^{\alpha\lambda}(\partial_\beta\partial_\lambda g_{\alpha\gamma} - \partial_\gamma\partial_\lambda g_{\alpha\beta}) + O((\partial g)^2).\end{aligned}$$

Finally we note that the remaining term other than $\widetilde{\Box}_g \; g_{\beta\gamma}$ is symmetric in β and γ and hence in the expression $2R_{\beta\gamma} = R_{\beta\gamma} + R_{\gamma\beta}$ it would be zero and so we conclude that

$$R_{\beta\gamma} = -\frac{1}{2}\widetilde{\Box}_g \; g_{\beta\gamma} + O((\partial g)^2).$$

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