HEIGHT FUNCTION ON DOMINO TILINGS

1. Summary

We first define the height function on a domino tiling (as done in [1]) and state some of its basic properties. We then revisit the coupling function and relate it to Green's function, which allows us to conclude that the coupling function converges in the limit to an analytic function with a pole. Using this, we do a general second moment calculation via the proof found in [1] which writes the moment in terms of integrals of functions which are conformally invariant.

2. Setup

We work with a simply connected domain U which has smooth boundary and mark some point d on ∂U. For $\epsilon > 0$, take P_ϵ to be a Temperleyan polynomino in $\epsilon\mathbb{Z}^2$ approximating U in such a way that ∂P_ϵ is within $O(\epsilon)$ of ∂U and $d_\epsilon \in P_\epsilon$ is within $O(\epsilon)$ of d, where d_ϵ is the removed square of P, and the counterclockwise boundary path of P_ϵ points locally into the same half–space as the tangent to ∂U to which it's near. (For technical reasons we also require that there is a segment of length δ on ∂P_ϵ which is straight such that δ tends to zero slowly compared to ϵ, but we shall not deal with "boundary problems" here.) Finally, the sentence "I is conformally invariant" means that if $f : U \to V$ is a conformal isomorphism, then

$$\int_\gamma I^u(z)dz = \int_{f(\gamma)} I^v(f(z))dz.$$

2.1 The Height Function.

Given a domino tiling on the usual setup of black and white squares (checkerboard tiling), the height function h is a function from the vertices of the domino tiling to $\mathbb{Z}$. The height function is defined up to an additive constant as follows: fix some vertex v_0 and declare the height there to be zero. Then for any other vertex v, take an edge path γ from v_0 to v which follows the boundaries of dominos (i.e. not allowed to cut dominos), then the height along γ changes by +1 if the edge traversed has a black square on its left and -1 if the edge traversed has a white square on its left (note that the black or white square here may very well not be inside the domain under consideration).

***Remark* 2.1.** *The height function is well–defined up to an additive constant, i.e. once the value at v_0 is fixed, the value at v is independent of the path γ we take. This is because the height change around any domino is equal to zero and by the discrete version of Green's Theorem, the height change along any loop is also zero and hence the height is path independent.*

***Remark* 2.2.** *First note that along each straight edge the height alternates, so along a straight edge of even length the height change is zero. Second, if a path does cross a domino, then the height change is +3 if when crossing the domino a black square is on the right and the height change is -3 otherwise.*

2.2 Coupling Function.

Recall that the coupling function is the inverse of the Kasteleyn matrix. For a fixed $v_1 \in W_0 \cup W_1$, we view $C(v_1, -)$ as a function defined on $B_0 \cup B_1$. So when $v_1 \in W_0$, the real part of $C(v_1, -)$ is defined on B_0 and the imaginary part is defined on B_1, whereas when $v_1 \in W_1$, the real part if defined on B_1 and imaginary part is defined on B_0. For what follows we will work with $B_0(P)$ (vertices of B_0 and edges between these vertices). We denote the external boundary of $B_0(P)$ by Y and let $B_0'(P) = B_0(P) \cup Y$. We extend the coupling function to be zero on Y.

3. Convergence of the Coupling Function

Theorem 3.1. *We now work on $\epsilon\mathbb{Z}^2$. Let v_1 and v_2 be far from the boundary and not within $o(1)$ of each other, then if $v_1 \in W_0$,*

$$\frac{1}{\epsilon}C(v_1, v_2) = \begin{cases} \mathrm{Re}F_0(v_1, v_2) + o(1) & \text{if } v_2 \in B_0, \\ iIMF_0(v_1, v_2) + o(1) & \text{if } v_2 \in B_1. \end{cases}$$

where F_0 is analytic as a function of z_2, has a simple pole of residue $1/\pi$ at $z_2 = z_1$, and has no other poles. Also, it is zero at d and has real part 0 on ∂U.

On the other hand, if $v_1 \in W_1$, then

$$\frac{1}{\epsilon}C(v_1, v_2) = \begin{cases} \mathrm{Re}F_1(v_1, v_2) + o(1) & \text{if } v_2 \in B_0, \\ iIMF_1(v_1, v_2) + o(1) & \text{if } v_2 \in B_1. \end{cases}$$

where F_1 is analytic as a function of z_2, has a simple pole of residue $1/\pi$ at $z_2 = z_1$, and has no other poles. Also, it is zero at d and has imaginary part 0 on ∂U.

Proof. We do the case $v_1 \in W_0$. The case $v_1 \in W_1$ is identical using the imaginary part of C.

Let $G(w_1, w_2)$ be the Green's function on $B_0'(P_\epsilon)$, i.e., $\triangle G(w_1, w_2) = \delta_{w_1}(w_2)$ and $G(w_1, w_2) = 0$ when w_2 is on the boundary. Now we have

$$\triangle \mathrm{Re} C(v_1, -) = \delta_{v_1+\epsilon} - \delta_{v_1-\epsilon} = \triangle(G(v_1+\epsilon, v_2) - G(v_1-\epsilon, v_2)).$$

Since C also vanishes on the boundary, we conclude that

$$\mathrm{Re} C(v_1, -) = G(v_1+\epsilon, v_2) - G(v_1-\epsilon, v_2).$$

We will first show that the real part converges, which implies convergence of its derivatives and hence by integrating convergence of $\mathrm{Im} C$ as well. Furthermore the function F_0 to which it converges is unique.

Lemma 3.2.

$$(1/\epsilon)G(v_1+\epsilon, v_1) - (1/\epsilon)(G(v_1-\epsilon, v_2) \to 2\partial_{x_1} g_u(z_1, z_2),$$

where g_u is the continuous Green's function on U with boundary value zero.

Proof. This is a standard argument so we will be brief. Set $H = (1/\epsilon)G(v_1+\epsilon, v_1) - (1/\epsilon)(G(v_1-\epsilon, v_2)$, and $H_0 = (1/\epsilon)G_0(v_1+\epsilon, v_1) - (1/\epsilon)(G_0(v_1-\epsilon, v_2)$, where H_0 and G_0 are the corresponding quantities for $\epsilon\mathbb{Z}^2$. First we have $H - H_0$ is harmonic in the second variable (since the poles subtract) and has bounded boundary values, since it is known that

$$H_0(v_1, v_2) = \mathrm{Re}\frac{1}{\pi(v_2 - v_1)} + O\left(\frac{1}{|v_2 - v_1|^2}\right).$$

Also the boundary values of $H - H_0$ are continuous in the limit as $\epsilon \to 0$. So we let g be the harmonic function with boundary value equal to this limiting boundary value. Next approximating the discrete Laplacian of g with the Taylor expansion of g, we find that $H - H_0 - G$ has discrete Laplacian of order $O(\epsilon^4)$. Finally, since $x + iy$ goes to x^2 has discrete Laplacian equal to a constant, we can pick constants A, B big enough so that the discrete Laplacians of both $A\epsilon^4\mathrm{Re}(v_2)^2 - (H - H_0 - g)(v_1, v_2)$ and $B\epsilon^4\mathrm{Re}(v_2)^2 + (H - H_0 - g)(v_1, v_2)$ are bigger than or equal to zero. Hence these functions are superharmonic and must achieve their maximum values on the boundary; since $\epsilon^4\mathrm{Re}(v_2)^2$ has order ϵ^2 and H_0 has order ϵ on the boundary, $|H - H_0 - g| = O(\epsilon)$ on the boundary and hence everywhere. So $H(v_1, v_2)$ converges to the function $\mathrm{Re}(\frac{1}{\pi}(v_2 - v_1)) + g(v_1, v_2)$, which has boundary value zero and pole of residue $1/\pi$ at v_1. This is exactly $2\partial_{x_1} g_u$. □

This completes the proof of the Theorem. □

Now let $F_+ = F_0 + F_1$ and $F_- = F_0 - F_1$. One can then show, using the fact that C is discrete analytic and passing to the limit, that F_+ is analytic also in the first variable and that F_- is antianalytic (i.e. $\partial_z(F_-) = 0$) in the first variable. Looking at the behavior of the pole and using the facts that F_0 and F_1 are unique, one sees also that F_+ and F_- also transform in a conformally invariant way under conformal isomorphisms.

4. Second Moment Calculation

Let x_1, x_2 be two interior points of U. Let $h(x_1)$, $h(x_2)$ denote the height at two points of P_ϵ within $O(\epsilon)$ of x_1 and x_2, respectively. We would like to compute

$$\mathbb{E}((h(x_1) - \overline{h})(h(x_2) - \overline{h})). \tag{4.1}$$

We take two disjoint paths γ_1 and γ_2 from x_1 and x_2 to the straight boundary near d (the height is defined to be zero there. In addition we require that each straight edge of γ_i has even length (choose parities of the relevant points so that this is possible).

Now in a given tiling, the height change along γ_i is equal to $4(A_{\gamma_i} - B_{\gamma_i})$, where A_{γ_i} is the number of dominos crossed with black squares on the right and B_{γ_i} is the number of dominos crossed with black square on the left: The straight edges are even so the height change along them is zero, and for each domino crossed, the height changes by +4 or -4, since if the black square is on the right and no domino is crossed then the change in height is -1 but if there is a domino crossed then the height change is +3, so the net change is +4; an identical argument shows the height change is -4 in the case of a domino crossing with black square on the left (J. Asher). So now (4.1) is equal to

$$4^2\mathbb{E}((A_{\gamma_1} - B_{\gamma_1} - (\overline{A_{\gamma_1}} - \overline{B_{\gamma_1}}))(A_{\gamma_2} - B_{\gamma_2} - (\overline{A_{\gamma_2}} - \overline{B_{\gamma_2}}))).$$

Let α_{jt} be the t^{th} possible domino crossed by γ_j whose black square is on the right and let β_{jt} be the t^{th} possible domino crossed by γ_j whose black square is on the left. We also let α and β denote the indicator functions of these events. Straight edge paths in γ_j are even so we can pair α_{jt} with adjacent $\beta_{jt'}$ which are parallel. Rewriting again, we now

have
(4.2)

$$\begin{aligned}
&\mathbb{E}((h(x_1)-\overline{h})(h(x_2)-\overline{h}))\\
&= 4^2\sum_{t,t'}\mathbb{E}[((\alpha_{1t}-\overline{\alpha_{1t}})-(\beta_{1t}-\overline{\beta_{1t}}))((\alpha_{2t}-\overline{\alpha_{2t}})-(\beta_{2t}-\overline{\beta_{2t}}))]\\
&\equiv 4^2\sum_{t,t'}\mathbb{E}((A_{1t}-B_{1t})(A_{2t}-B_{2t}).
\end{aligned}$$

Now from the theorem on perfect matchings, if we let $e_i = (w_i, b_i)$, then

$$\mathbb{E}((e_1-\overline{e_1})(e_2-\overline{e_2}) = -a_E C(w_1,b_2)C(w_2,b_1),$$

where a_E is the product of the edge weights of e_1 and e_2. (A similar formula holds for more than two terms: We would get the edge weights times the matrix corresponding to $\mathbb{E}(e_1,\dots,e_n)$ with all zeroes on the diagonal.)

Now observe that each β edge has weight of the opposite sign as the α edge to which it is paired, so when one expands out the the summand in (4.2), the signs cancel the sign changes in a_E so that we are left with terms of the form $C(w_1,b_2)C(w_2,b_1)$ times the edge weight a_E of the α edges.

Next if we let $r_i = \pm 1$ according to whether $w_i \in W_0$ or $w_i \in W_1$ and $s_i = \pm 1$ according to whether $b_i \in B_0$ or $b_i \in B_1$, then from Theorem 3.1, we can write (in the good cases where nobody is near the boundary; the bad cases we refuse to think about)

$$\begin{aligned}
&C(w_1,b_2)\\
&= \epsilon\left(\frac{1-r_1s_2}{2}i\mathrm{Im}+\frac{1+r_1s_2}{2}\mathrm{Re}\right)\left(\frac{1+r_1}{2}F_0(w_1,b_2)+\frac{1-r_1}{2}F_1(w_1,b_2)\right)\\
&= \frac{\epsilon}{4}(F_+(w_1,b_2)+r_1F_-(w_1,b_2)+s_2\overline{F_-}(w_1,b_2)+r_1s_2\overline{F_+}(w_1,b_2))+o(\epsilon).
\end{aligned}$$

So if we let $\alpha_{it} = (w_{it}^\alpha, b_{it}^\alpha)$ and $\beta_{it} = (w_{it}^\beta, b_{it}^\beta)$, then we have, for example,
(4.3)

$$\begin{aligned}
\mathbb{E}(A_{1t}A_{2t'}) = \frac{-a_E\epsilon^2}{4^2}&(F_+ + r_1F_- + r_1s_2\overline{F_+} + s_2\overline{F_-})(w_{1t}^\alpha, b_{2t'}^\alpha)\\
&\times (F_+ + r_2F_- + r_2s_1\overline{F_+} + s_1\overline{F_-})(w_{2t'}^\alpha, b_{1t}^\alpha) + o(\epsilon^2).
\end{aligned}$$

Now in the limit w_{1t}^α, w_{1t}^β, b_{1t}^α and b_{1t}^β will all go to z_t since they are all within ϵ of each other and similarly the t' vertices will go to $z_{t'}$. We will have four terms like those in (4.3), but observe that if we go from $(w_{it}^\alpha, b_{it}^\alpha)$ to $(w_{it}^\beta, b_{it}^\beta)$, then the sign of r_i and s_i reverse, so in the limit when we replace the relevant vertices by z_t and $z_{t'}$, we will have cancellations after which we are left only with 4 times the terms with r_i to the same power as s_i, so we have the summand in (4.2) as

$$\frac{-a_E\epsilon^2}{4}\sum_{\epsilon_1,\epsilon_2\in\{-1,1\}}(r_1s_1)^{\frac{1-\epsilon_1}{2}}(r_2s_2)^{\frac{1-\epsilon_2}{2}}F_{\epsilon_1,\epsilon_2}(z_t,z_{t'})F_{\epsilon_2,\epsilon_1}(z_{t'},z_t),$$

where $F_{1,1}=F_+$, $F_{-1,1}=F_-$, $F_{1,-1}=\overline{F_-}$ and $F_{-1,-1}=\overline{F_+}$.

The final step is to replace the sum over t and t' by integrals. For this, we replace 2ϵ by some phase times dz_τ or $\overline{dz_\tau}$, $\tau=t,t'$. For example, if the path is going east, then $2\epsilon=dx_t=dz_t=\overline{dz_t}$, the α edge has weight $-i$ and $r_js_j=-1$ (here we are crossing a vertical domino, so if the black square is type B_1 then the white square is of type W_0 and vice versa). Similarly we can find the corresponding values when the path is going west, north or south. In all cases,

$$2\epsilon\times a_E\times(r_js_j)^{1-\epsilon_j/2}=-\epsilon_j i dz_\tau^{(\epsilon_j)},$$

where $dz^{(1)}=dz$ and $dz^{(-1)}=\overline{dz}$.

Finally, constant factors of 4 cancel, $(-i)^2$ cancels the overall negative sign in front, ϵ goes to the corresponding dz_τ, sum over t and t' gets replaced by integrals and we are left with

$$\lim_{\epsilon\downarrow 0}\mathbb{E}((h(x_1)-\overline{h})(h(x_2)-\overline{h}))$$
$$=\sum_{\epsilon_1,\epsilon_2\in\{\pm1\}}\epsilon_1\epsilon_2\int_{\gamma_1}\int_{\gamma_2}F_{\epsilon_1,\epsilon_2}(z_1,z_2)F_{\epsilon_2,\epsilon_1}(z_2,z_1)dz_t^{(\epsilon_1)}dz_{t'}^{(\epsilon_2)}.$$

***Remark* 4.1.** *When there are more than two terms involved, we would have to sum over fixed–point free permutations (fixed point free since there is no diagonal term) when taking the determinant to find* $\mathbb{E}((\alpha_1-\overline{\alpha_1})\dots(\alpha_n-\overline{\alpha_n})$ *and we would in fact get* $\det(F_{\epsilon_i,\epsilon_j}(z_i,z_j))$ *in the integrand.*

References

[1] R. Kenyon. *Conformal Invariance of Domino Tilings.* Annals of Probability. **28**, no. 2, 759-795 (2000).

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