Finite Element Method for PDEs

Helen K. Lei

September 22, 2007

1 Poincaré Type Inequalities

Typically we have a variational formulation of the form find $u \in V$ such that

$$A(u, v) = L(v), \forall v \in V,$$

where V is some appropriate space of functions (e.g. $H^1(\Omega)$), A is bilinear and L is linear. To show the variational problem is well–posed (i.e. has a unique solution) we need to appeal to the Lax–Milgram Lemma, which takes as one of the hypotheses that A is continuous and elliptic, i.e. $\|A(u, v)\|_V \leq C\|u\|_V\|v\|_V$ and $A(u, u) \geq C'\|u\|_V^2$, where $\|\cdot\|_V$ denotes the norm on V and C and C' are constants.

In showing ellipticity of the bilinear form associated with a variational formulation, we often need to appeal to Poincaré type inequalities which relate the norm of the function to the norm of its derivative. The simplest such inequality is for a function $f \in H_0^1([0, a])$, that is

$$f : [0, 1] \to \mathbb{R} \text{ such that } f, f' \in L^2([0, a]) \text{ and } f(0) = f(a) = 0.$$

Let $t \in (0, a)$, then by the Fundamental Theorem of Calculus and the boundary condition $f(0) = 0$, we have

$$f(t) = \int_0^t f'(s)\ ds.$$

Applying the Cauchy–Schwarz inequality, we immediately obtain

$$\begin{aligned} |f(t)| &\leq \left(\int_0^t 1\ ds\right)^{1/2} \left(\int_0^t |f'(t)|^2\ ds\right)^{1/2} \\ &\leq \sqrt{a}\|f'\|_{L^2([0,a])}, \end{aligned}$$

where to obtain the second inequality we have replaced the upper limit of integration by a. Now integrating $|f(t)|^2$, we find

$$\int_0^a |f(t)|^2 \, ds \leq \int_0^a a\|f'\|^2_{L^2([0,a])} \, ds$$
$$= a^2\|f'\|^2_{L^2([0,a])}.$$

From which we conclude

$$\|f\|_{L^2([0,a])} \leq a\|f'\|_{L^2([0,a])}.$$

We also see that since $\|f\|_{H_0^1([0,a])} = \left(\int_0^a |f(t)|^2 + |f'(t)|^2 \, ds\right)^{1/2}$,

$$\|f\|_{H_0^1} \leq \sqrt{a^2+1}\|f'\|_{L^2}.$$

Since it is clear that

$$\|f'\|_{L^2} \leq \|f\|_{H_0^1},$$

we conclude that $\|f'\|_{L^2}$ and $\|f\|_{H_0^1}$ are norm equivalents. Observe that the constants involved depend on the geometry of the domain. For convenience let's make the following definition

Definition 1.1. Let Ω be a bounded open set. We let

$$|f|_{H^1(\Omega)} = \|\nabla f\|_{L^2(\Omega)}.$$

The above can be generalized and we in fact obtain the following

Theorem 1.2. *Let Ω be a bounded open set. Then there exists constant C_Ω such that for all $v \in H_0^1(\Omega)$*

$$C_\Omega\|v\|_{L^2(\Omega)} \leq |v|_{H^1(\Omega)}.$$

Here of course $H_0^1(\Omega)$ consists of all functions in $H^1(\Omega)$ which vanish on the boundary. (The L^p analogue of the above is also true for $1 \leq p < \infty$, but we shall not need that here). We sometimes need this theorem when showing ellipticity of the bilinear form associated with the variational formulation of a problem with Dirichlet boundary conditions, as in this case, the natural space of functions to use is H_0^1.

But we can do a little better than this. First notice that we can easily generalize the first derivation to functions $f \in H_0^1([0,a] \times [0,b])$ by applying the fundamental theorem of calculus to say $\frac{\partial f}{\partial x}$ and integrating from say some point on the left boundary. The point is the function need not vanish on the entire boundary, only some part of the boundary. More precisely, we have the following

Theorem 1.3. *Let Ω be a bounded connected open set with sufficiently regular boundary Γ. Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ with length of $\Gamma_2 > 0$. Let*

$$V_{\Gamma_2} = \{u \in H^1(\Omega) : u|_{\Gamma_2} = 0\}.$$

Then V_{Γ_2} is a closed subspace of $H^1(\Omega)$ and

$$|v|_{H^1(\Omega)} \sim \|u\|_{H^1(\Omega)} \text{ on } V_{\Gamma_2},$$

where $\sim$ denotes norm equivalence.

This theorem is needed sometimes when we have a mixed Dirichlet–Neumann problem where on one part of the boundary the function is specified whereas on the remaining part information about the normal derivative is given.

Let us return once again to our first calculation. Now suppose $f \in H^1((0,a))$, but instead of assuming f vanishes at 0, assume that

$$\int_0^a f(t)\ dt = 0.$$

Now we have

$$f(t) = \int_0^t f'(s)\ ds + f(0),$$

so

$$\begin{aligned}
(f(t))^2 &= \left(\int_0^t f'(s)\ ds\right)^2 + 2f(0)\int_0^t f'(s)\ ds + (f(0))^2 \\
&= \left(\int_0^t f'(s)\ ds\right)^2 + 2f(0)(f(t) - f(0)) + (f(0))^2 \\
&= \left(\int_0^t f'(s)\ ds\right)^2 + 2f(0)f(t) - (f(0))^2 \\
&\leq \left(\int_0^t f'(s)\ ds\right)^2 + 2f(0)f(t) \\
&\leq a|f|^2_{H^1((0,a))} + 2f(0)f(t),
\end{aligned}$$

where to obtain the last inequality we have again changed the upper limit of integration to a and applied the Cauchy–Schwarz inequality. Integrating, we now obtain

$$\begin{aligned}
\|f\|^2_{L^2([0,a])} &\leq a^2|f|^2_{H^1((0,a))} + 2f(0)\int_0^a f(t)\ dt \\
&= a^2|f|^2_{H^1((0,a))}.
\end{aligned}$$

Notice the condition $\int_0^a f(t)\ dt = 0$ has been used to get rid of the second term. Taking square roots, we obtain as before that

$$\|f\|_{L^2([0,a])} \leq a|f|_{H^1((0,a))}.$$

This can be generalized and we obtain

Theorem 1.4. *Let Ω be a bounded connected open set with sufficiently regular boundary. Let*

$$V_0 = \{u \in H^1(\Omega) : \int_\Omega u(x)\ dx = 0\}.$$

Then V_0 is closed in $H^1(\Omega)$ and

$$|u|_{H^1(\Omega)} \sim \|u\|_{H^1(\Omega)} \text{ on } V_0.$$

The condition $\int_\Omega u(x)\ dx = 0$ is a natural condition to impose in a Neumann problem, which has boundary condition of the form $\frac{\partial u}{\partial n} = g$ on Γ, as otherwise the solution is not uniquely determined.

Next we would like to derive one more inequality which bounds the L^2 norm over the boundary by the H^1 norm in the interior. Again let $f \in H^1((0,a))$. If $f \in (0,a)$, then by the Fundamental Theorem of Calculus,

$$f(0) = f(t) - \int_0^t f'(s)\ ds,$$

so that

$$\begin{aligned}
(f(0))^2 &= (f(t))^2 - 2f(t)\int_0^t f'(s)\ ds + \left(\int_0^t f'(s)\ ds\right)^2 \\
&\leq |f(t)|^2 + 2\int_0^t |f(t)||f'(s)|\ ds + \left(\int_0^t |f'(s)|\ ds\right)^2.
\end{aligned}$$

Letting the upper limit of integration by a and applying the Cauchy–Schwarz inequality to the last two terms, we get

$$\begin{aligned}
(f(0))^2 &\leq |f(t)|^2 + 2\sqrt{a}|f(t)||f|_{H^1} + a|f|_{H^1}^2 \\
&\leq (|f(t)| + \sqrt{a}|f|_{H^1})^2.
\end{aligned}$$

Similarly,

$$f(a) = f(t) + \int_0^t f'(s)\ ds$$

and

$$(f(a))^2 \leq (|f(t)| + \sqrt{a}|f|_{H^1})^2.$$

So that

$$(f(0))^2 + (f(a))^2 \leq 2(|f(t)| + \sqrt{a}|f|_{H^1})^2$$

and

$$\sqrt{(f(0))^2 + f(a))^2} \leq \sqrt{2}(|f(t)| + \sqrt{a}|f|_{H^1}).$$

Finally we integrate this over t and apply the Cauchy–Schwarz inequality two more times, to obtain

$$\begin{aligned} a\sqrt{(f(0))^2 + f(a))^2} &\leq 2\int_0^a (|f(t)| + \sqrt{a}|f|_{H^1})\ dt \\ &\leq 2(\sqrt{a}\|f\|_{L^2} + a|f|_{H^1}). \end{aligned}$$

Now note that

$$\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + |f|_{H^1}^2} \geq \frac{1}{2}(\|f\|_{L^2} + |f|_{H^1}),$$

where the last inequality come from the elementary inequality that $\alpha^2 + \beta^2 \geq \frac{1}{2}(\alpha + \beta)^2$, since $(\alpha - \beta)^2 \geq 0$ so that $\alpha^2 + \beta^2 \geq 2\alpha\beta$. So using this, we finally obtain

$$\sqrt{(f(0))^2 + f(a))^2} \leq 4\max\{\sqrt{a}, a\}\|f\|_{H^1((0,a))}.$$

The general theorem is as follows

Theorem 1.5. *Let Ω be an open and bounded domain with Lipschitz–continuous boundary $\Gamma = \partial\Omega$. Then there is a constant C such that*

$$\|u|_\Gamma\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

This is needed in showing continuity of the bilinear form associated with Robin's problem, which has boundary condition of the form $\gamma u + \frac{\partial u}{\partial n} = g$ on Γ, with $\gamma > 0$. In showing the ellipticity of the bilinear form associated with Robin's problem, since the space of functions we consider may not vanish on a significant piece of the boundary and we do not have the normalization condition $\int_\Omega u(x)\ dx = 0$, we will need the more general Poincaré's inequality

Theorem 1.6. *Let Ω be a bounded connected open set with sufficiently smooth boundary Γ. Then there is some constant C such that*

$$\int_\Omega |u(x)|^2\ dx \leq C\left(\int_\Omega |\nabla u|^2\ dx + \int_\Gamma |u(s)|^2\ ds\right).$$

Again in the one dimensional case, this is derived by writing e.g.

$$f(t) = \int_0^t f'(s)\ ds + f(0),$$

squaring, integrating and using the Cauchy–Schwarz inequality.

2 Error Estimates

We approximate u by solving the variational problem in some finite element space $V_h \subset V$, which is a finite dimensional vector space. Usually we take a triangulation of the domain Ω and then for example if $V = H^1(\Omega)$, then we can take the piecewise linear space

$$V_h = \{v : v|_K \text{ is linear, for all triangles } K \text{ in the triangulation}$$
$$\text{and } v \text{ is continuous at the nodes of the triangulation}\}.$$

We then look for $u_h \in V_h$ such that

$$A(u_h, v) = L(v), \forall v \in V_h.$$

We are then interested in the error

$$\|u - u_h\|_V,$$

where $\|\cdot\|_V$ denotes the norm on V. We first note that we have the abstract error estimate that there is some constant C such that

$$\|u - u_h\|_V \leq C\|u - v\|_V, \forall v \in V_h.$$

So we may obtain an estimate on $\|u - u_h\|$ by considering a suitable function $v \in V_h$ and estimating $\|u - v\|$. Usually we take

$$v = \pi_h u,$$

where $\pi_h u$ is an interpolant of u in V_h. E.g. in the piecewise linear case, $\pi_h u$ would agree with u on the nodes of the triangulation. We have the following estimates for the interpolant error: If we use piecewise polynomials of degree $r \geq 1$ on triangulation T_h satisfying some regularity assumption, then

$$\|u - \pi_h u\|_{L^2} \leq Ch^{r+1}|u|_{H^{r+1}},$$
$$\|u - \pi_h u\|_{H^1} \leq Ch^{r}|u|_{H^{r+1}}$$

and

$$\|u - \pi_h u\|_{H^2} \leq Ch^{r-1}|u|_{H^{r+1}}.$$

To state the regularity assumption on the triangulation, let

$$h_K = \text{diameter of } K = \text{longest side of } K$$

$$\rho_K = \text{diameter of the circle inscribed in } K$$

and $h = \max_{K \in T_h} h_K$, then there must be some constant β which is independent of h such that for any K in the triangulation T_h (for this family of triangulations $\{T_h\}$)

$$\frac{\rho_k}{h_K} \geq \beta,$$

that is, no triangles are not allowed to be arbitrarily thin.

3 General Description

Type of problems we consider (elliptic PDEs), variational/minimum formulation, ellipticity, function spaces, Lax-Milgram, abstract error estimate; linear system: ellipticity implies positive definiteness/nonsingularity, error estimate of interpolant; rate of convergence

4 Variational Formulation

Green's Formula, (continuous) choice of space, natural/essential boundary (Dirichlet/Neumann); Poincare type inequalities

5 Finite Element Spaces in 2D

Regularity and (discrete) choice of space, degrees of freedom, P_1, P_2, P_3, P_5, error estimates; linear system: sparse, positive definiteness/nonsingularity

6 The Biharmonic Problem