List of Common Finite Difference Schemes

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1 Hyperbolic

- Forward–time forward space (explicit, one–step, order $(1, 1)$, stable if $-1 \leq a\lambda \leq 0$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = 0$$

- Forward–time backward–space (explicit, one–step, order $(1, 1)$, stable if $0 \leq a\lambda \leq 1$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0$$

- Forward–time central–space (explicit, one–step, order $(1, 2)$, unstable):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

- Lax–Friedrichs (explicit, one–step, order $O(k^{-1}h^2) + O(k) + O(h^2) = O(h)$ for $k = \lambda h$, stable if $|a\lambda| \leq 1$):

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

- Backward–time forward–space (implicit, one–step, order $(1, 1)$, stable if $a \leq 0$):

$$\frac{v_m^n - v_m^{n-1}}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = 0$$

- Backward–time backward–space (implicit, one–step, order $(1, 1)$, stable if $a \geq 0$):

$$\frac{v_m^n - v_m^{n-1}}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0$$

- Backward–time central–space (implicit, one–step, order (1, 2), unconditionally stable):

$$\frac{v_m^n - v_m^{n-1}}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

- Leapfrog (explicit, multi–step, order (2, 2), stable if $|a\lambda| < 1$):

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

- Lax–Wendroff (Explicit, one–step, order (2, 2), stable if $|a\lambda| \leq 1$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2 k}{2}\frac{(v_{m+1}^n - 2v_m^n + v_{m-1}^n)}{h^2} = \frac{1}{2}(f_m^{n+1} + f_m^n) - \frac{ak}{4h}(f_{m+1}^n - f_{m-1}^n)$$

- Crank–Nicolson (Implicit, one–step, order (2, 2), unconditionally stable):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = \frac{f_m^{n+1} + f_m^n}{2}$$

2 Parabolic

- Forward–time central–space (Explicit, one–step, order (1, 2), stable in both $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $b\mu \leq 1/2$):

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

- Backward–time central–space (implicit, one–step, order (1, 2), unconditionally stable in $\|\cdot\|_2$ and $\|\cdot\|_\infty$):

$$\frac{v_m^n - v_m^{n-1}}{k} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

- Crank–Nicolson (implicit, one–step, order (2, 2), unconditionally stable in $\|\cdot\|_2$ and stable in $\|\cdot\|_\infty$ if $b\mu \leq 1$):

$$\begin{aligned}\frac{v_m^{n+1} - v_m^n}{k} &= \frac{1}{2}b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} \\ &\quad + \frac{1}{2}b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} + \frac{1}{2}(f_m^{n+1} + f_m^n).\end{aligned}$$

- Du Fort–Frankel (explicit, multi–step, order $O(h^2) + O(k^2) + O(k^2h^{-2}) = O(h^2)$, for $k = h^2$, unconditionally stable):

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b\frac{v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n}{h^2} + f_m^n$$

3 Higher Order

- Central–time central–space (explicit, two–step, order $(2, 2)$, stable if $a\lambda \leq 1$ for $a \geq 0$):

$$\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} = a^2 \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

4 Convection–Diffusion

- Forward–time central–space (explicit, one–step, order $(2, 2)$, stable in $\|\cdot\|_2$ if $b\mu \leq 1/2$, stable in $\|\cdot\|_\infty$ if $h \leq \frac{2b}{a}$):
- Upwind differencing $(a > 0)$ (Order $(1, 2)$, stable in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $a\lambda + 2b\mu \leq 1$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

- Upwind differencing $(a < 0)$ (Order $(1, 2)$, stable in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $|a|\lambda + 2b\mu \leq 1$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

5 ADI: Peaceman–Rachford

- Peaceman–Rachford (Order $(1, 2)$, unconditionally stable):

$$\left(I - \frac{k}{2}A_{1h}\right)\tilde{v}^{n+1/2} = \left(I + \frac{k}{2}A_{2h}\right)v^n$$
$$\left(I - \frac{k}{2}A_{2h}\right)v^{n+1} = \left(I + \frac{k}{2}A_{1h}\right)\tilde{v}^{n+1/2}$$