Some Notes on *The Geometry of Dissipative Evolution Equations: The Porous Medium Equation* by Felix Otto

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1 Introduction and Gradient Flows

We think of $\rho \geq 0$ as a density on $\mathbf{R}^N$ and study the equation

$$\frac{\partial \rho}{\partial t} - \nabla^2 \rho^m = 0,$$

with $m \geq 1 - \frac{1}{N}$ and $m > \frac{N}{N+2}$. By the chain rule, we may write this as

$$\frac{\partial \rho}{\partial t} = m\rho^{m-1}\nabla^2 \rho + m(m-1)\rho^{m-2}|\nabla \rho|^2,$$

and hence

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [a(\rho)\nabla \rho],$$

where $a(\rho) = m\rho^{m-1}$. We point out three properties of the porous medium equation.

- (Finite Propagation) Note that $a(\rho)$ vanishes at $\rho = 0$; this leads in particular for $m > 1$ to the finite propagation property (see [6]): Making the change of variables

  $$\hat{\rho} = \frac{m}{m-1}\rho^{m-1},$$

  a direct computation shows that

  $$\hat{\rho}_t = (m-1)\hat{\rho}\nabla^2 \hat{\rho} + |\nabla \hat{\rho}|^2,$$

  so for $\hat{\rho}$ small, we can approximate by

  $$\hat{\rho}_t = |\nabla \hat{\rho}|^2,$$

  which is bounded if ρ is smooth. Phrased differently, the evolution given by the porous medium equation preserves compact support if $m > 1$.

- (Preservation of Mass) Let $0 \leq \zeta \leq 1$ be a smooth cutoff function, then two applications of the divergence theorem gives

  $$\frac{d}{dt}\int \rho_t \zeta \; dx = \int (\nabla^2 \rho_t^m)\zeta \; dx = \int \rho_t^m (\nabla^2 \zeta) \; dx.$$

Here the spatial domain is $\mathbf{R}^d$ and we assume ρ to be integrable and tending to 0 at infinity, so taking $\zeta \to 1$, we see that $\int \rho$ is preserved [7].

- (Preservation of Positivity) This should follow from the maximum principle (see e.g. [1]) and/or note the linear approximation for small $\hat{\rho}$ from the previous item: $\hat{\rho}_t = |\nabla \hat{\rho}|^2$.

Our goal would be to interpret the evolution given by the equation as gradient flow on some manifold and derive some rigorous asymptotic results about its convergence to the Barenblatt solution. In $\mathbf{R}^n$, given a vector field X (which may or may not be autonomous – time independent), we may define the flow associated with X as $\Phi : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying the system of ODE's

$$\Phi(0,x) = x, \quad \dot{\Phi}(t,z) = X(t, \Phi(z,t)).$$

A good example to keep in mind is perhaps Hamiltonian flow. Given some $H : \mathbf{R}^{2n} \to \mathbf{R}$ corresponding to the Hamiltonian (total energy) of some system, Hamilton's equations say that $\frac{\partial H}{\partial p} = \dot{q}$, $\frac{\partial H}{\partial q} = -\dot{p}$, where q, p are position and momentum, respectively. Hence if we define $V = J\nabla H$, where $J = \begin{pmatrix} 0 & -\mathbf{I}_d \\ -\mathbf{I}_d & 0 \end{pmatrix}$, then the flow associated with V describes where we end up (in position–momentum space with the dimension of the physical space being n) starting at x and evolving according to Hamilton's equations. Suppose now instead of this "deterministic" situation, which is described by flows on $\mathbf{R}^d$, we would like to start off with some initial probability density and ask how the density function evolves. This leads to the notion that we should seek to study flows on more general objects, i.e. manifolds.

Since the porous medium equation preserves mass and positivity, as a first approximation, we can consider the manifold

$$\mathcal{M} = \{\rho \geq 0 : \int \rho = 1\}.$$

Eventually, of course, we would be led to study the flow of probability *measures*, but first we shall do things formally and this $\mathcal{M}$ will suffice for us. First (for linguistic purposes if nothing else), we gather some facts about manifolds.

2 Gradient Flow on a Riemannian Manifold

A manifold is a topological space $\mathcal{M} \subset \cup_\alpha U_\alpha$ equipped with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^d$, where V_α is open in $\mathbf{R}^d$. We require $\varphi_\alpha \circ \varphi_\beta^{-1}$ to be smooth.

The tangent space at a point $p \in M$, denoted T_pM, consists of all vectors v which can be represented as $\gamma'(0)$ where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve such that $\gamma(0) = p$, so e.g. $T_x\mathbf{R}^d$ is in fact all of $\mathbf{R}^d$, for any $x \in \mathbf{R}^d$; on the other hand, if $M = \mathbf{S}^d$, a sphere in $d+1$ dimensions, then the tangent space at $x \in \mathbf{S}^d$, is (homeomorphic to) $\mathbf{R}^d$. We can view an element of $T_p(M)$ as a function from $C^\infty(M)$ to $\mathbf{R}$ which takes the directional derivative of $f \in C^\infty(M)$ at the point p.

More precisely, a linear map $X : C^\infty(M) \to \mathbf{R}$ is called a *derivation* at p if it satisfies the Leibnitz rule, i.e.

$$X(fg) = f(p)Xg + g(p)Xf, \quad \forall f, g \in C^\infty(M).$$

The tangent vectors as described above satisfy this rule and alternatively, the tangent space $T_p(M)$ can be defined to be the set of all derivations at p. On $\mathbf{R}^d$, each vector v gives rise to some linear functional (element of $C^\infty(M)^*$) at each point $x \in \mathbf{R}^d$

$$D_v|_x : C^\infty(\mathbf{R}^d) \to \mathbf{R} : f \mapsto (D_v f)(x) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}.$$

This is nothing other than the directional derivative in the direction v evaluated at x. $D_v|_x$ is a derivation and using the basis representation $D_v|_x = v^i \frac{\partial}{\partial x^i}$ and Taylor's Theorem, we can show that $v_x \mapsto D_v|_x$ is an isomorphism between the space of derivations at x and $\mathbf{R}^n_a$.

We now move from the above "pointwise" description to a more global picture: Suppose we want to take the directional derivative of a function f but in a different direction depending on where we are. This leads to the notion of the *tangent bundle* $TM = \uplus_{p\in M} T_p M$, and a (smooth) *vector field*, which is a *section* of TM

$$X : M \to TM : p \mapsto X(p) \in T_p M.$$

A vector field then defines an operation on $C^\infty(M)$:

$$X : C^\infty(M) \to C^\infty(M) : f \mapsto Xf : Xf(p) = X(p)f|_p,$$

which we may intuitive interpret as taking the directional derivative of f in the direction $X(p)$ at the point p. On the other hand, a vector field can be multiplied by $f \in C^\infty(M)$

$$X \mapsto fX : p \mapsto f(p)X(p) \in T_p M,$$

and hence the set of all vector fields, denoted $\Gamma(\mathcal{M})$, is in fact a module over $C^\infty(\mathcal{M})$ with the natural linear structure.

Next we must discuss the notion of gradient. We have that $f \in C^\infty(\mathbf{R}^d)$ defines a natural gradient vector field by $f \mapsto \nabla f = \sum_i \frac{\partial f}{\partial x_i}\partial_i$, where we have taken $\{\partial_1, \ldots, \partial_d\}$ as a basis for the tangent space. However, this is clearly not a coordinate–independent expression: E.g. consider what happens in $\mathbf{R}^2$ when we change from Cartesian to polar coordinates – we do not obtain the gradient as $\frac{\partial f}{\partial r}\partial_r + \frac{\partial f}{\partial \theta}\partial_\theta$ in general, and hence this is not the correct notion to be generalized to the manifold setting.

A change of perspective leads to a better object. Recall that a vector field acts on $C^\infty(M)$ by sending f to Xf. Let us now reverse the role of f and X: We may instead say that $f \in C^\infty(M)$ defines a map on $\Gamma(M)$, by

$$f \mapsto df : \Gamma(M) \to C^\infty(M) : X \mapsto df(X) = Xf.$$

Restricting attention to each fiber of TM (i.e. going back to a pointwise point of view), we see that we have

$$f \mapsto df_p : T_p M \to \mathbf{R} : X_p \mapsto df_p(X_p) = X(p)f|_p.$$

That is, we see that df_p is a linear functional on $T_p M$, i.e. $df_p \in T_p^* M$ and is hence a *cotangent vector*, and so df is a covector field. df will turn out to be none other than the calculus notion of differential:

$$df = \frac{\partial f}{\partial x_i} dx_i$$

and is invariant under coordinate change. To see this, recall that our basis for the tangent space is $\{\partial_1, \ldots, \partial_d\}$; in addition let us take $\{dx^1, \ldots, dx^d\}$ to be a basis for the cotangent space, where, of course, we have $dx^i(\partial_j) = \delta_{ij}$. Let us write $df = \alpha_j dx^j$ and compute:

$$df(\partial_i) = \alpha_j dx^j(\partial i) = \alpha_i,$$

but $df(\partial_i)$ is by definition nothing other than $\frac{\partial f}{\partial x_i}$. To be consistent with Otto's notation, we will denote

$$df_p(X) = \mathrm{diff}\, f|_p.X$$

For a nice exposition on manifolds see [5]; a good book is [2].

What we say about Riemannian manifolds are from (in addition to Otto's paper) [4] and a course by P. Petersen in Fall 2008. A Riemannian manifold is a manifold M equipped with a smooth Euclidean inner product g_p on each T_pM, i.e. if X and Y are smooth vector fields, then $g_p(X|_p, Y|_p)$ is a smooth function of p. The metric allows us to recover a notion of gradient as follows: If $f \in C^\infty(M)$, we *define* $\mathrm{grad}\, f$ to be the vector field satisfying

$$g(X, \mathrm{grad}\, f) = df(X), \quad \forall X \in TM.$$

Since T_pM is a vector space, by the Riez Representation Theorem (at least for the finite dimensional case), $\mathrm{grad}\, f$ is uniquely defined. The gradient clearly depends on the metric g, and by our remarks from the previous section, we see that it is not possible to define an invariant notion of gradient without g (see [4], Section 1.1.1 on page 22).

A gradient flow of E on a Riemannian manifold $(\mathcal{M}, g)$ is given by the differential equation

$$\frac{d\rho}{dt} = -\mathrm{grad}\, E_{|\rho}.$$

Here $E : \mathcal{M} \to \mathbf{R}$ is thought of as some energy functional. Notice that taking dot product with g, we in fact have

$$g_\rho\left(\frac{d\rho}{dt}, s\right) + \mathrm{diff}\, E_{|\rho}.s = 0,$$

for all vector fields s along ρ. Notice that with $s = \frac{d\rho}{dt}$, we have

$$\frac{d}{dt}E(\rho) = \mathrm{diff}\, E_{|\rho}.\frac{d\rho}{dt} = -g_\rho\left(\frac{d\rho}{dt}, \frac{d\rho}{dt}\right),$$

so that the energy is decreasing along ρ_t. Note that to see the first equality we compare the expressions

$$\mathrm{diff}\, E_{|\rho}.\frac{d\rho}{dt} = \lim_{h\to 0} \frac{E(\rho + h\frac{d\rho}{dt}) - E(\rho)}{h} \quad \text{and} \quad \frac{d}{dt}E(\rho)_{|t} = \lim_{t\to 0} \frac{E(\rho(t+h)) - E(\rho(t))}{t}$$

and Taylor expand.

3 Two Interpretations of the Porous Medium Equation as Gradient Flow

Recall we are working with the manifold

$$\mathcal{M} = \{\rho \geq 0 : \int \rho = 1\}.$$

We will think of the tangent space as

$$T_\rho \mathcal{M} = \{s : \int s = 0\},$$

the set of mean zero functions. We will not elaborate on this precisely, except to remark that 1) in a natural sense, mean zero functions are orthogonal to constant functions; 2) the role of tangent vectors is to enable us to take derivatives and hence we will consider e.g. $E(\rho+s)$ where $s \in T_\rho M$ and since the evolution preserves mass, we are forced to use mean zero functions in the tangent space; 3) a function f can usually be written as $f = f_0 + f_c$, where f_0 is a mean zero function and f_c is a constant function. Thus, from this perspective, we are viewing the tangent space as a quotient space where we "divide" out by constant functions (i.e. identify functions which differ by a constant in some way). This identification will be done via elliptic equations.

3.1 Traditional Approach

In the traditional approach, to obtain $T_\rho \mathcal{M}$, given some mean zero s, we identify all functions q such that

$$-\nabla^2 q = s.$$

We define the metric tensor by

$$g_\rho(s_1, s_2) = \int \nabla q_1 \cdot \nabla q_2.$$

g_ρ is well defined since *assuming* $\int \nabla \cdot (q_1 \nabla q_2) = 0$, we see that

$$g_\rho(s_1, s_2) = \int s_1 q_2 = \int s_2 q_1,$$

so that e.g. if $-\nabla^2 \tilde{q}_1 = \tilde{s}_1 = s_1$ (where $\tilde{q}_1$ is possibly different from q_1), then

$$g_\rho(\tilde{s}_1, s_2) = \int s_2 \tilde{q}_1 = \int \tilde{s}_1 q_2 = \int s_1 q_2 = g_\rho(s_1, s_2).$$

The energy functional is given by

$$E(\rho) = \frac{1}{m+1} \int \rho^{m+1}.$$

Let us compute the differential of E along s evaluated at ρ:

$$\begin{aligned}
\operatorname{diff} E(\rho).s &= \lim_{t\to 0} \frac{E(\rho+ts)-E(\rho)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\frac{1}{m+1}\int [(\rho+ts)^{m+1}-\rho^{m+1}] \\
&= \int \lim_{t\to 0} \frac{1}{t}\left[\frac{\rho^{m+1}+(m+1)\rho^m ts + o(t^2) - \rho^{m+1}}{m+1}\right] \\
&= \int \rho^m s.
\end{aligned}$$

(This is the Gâteaux differential.) The differential equation $g_\rho\left(\frac{d\rho}{dt}, s\right) + \operatorname{diff} E(\rho).s = 0$ then becomes

$$\begin{aligned}
0 &= \int \frac{d\rho}{dt}q + \rho^m s \\
&= \int \frac{d\rho}{dt}q - \rho^m \nabla^2 q \\
&= \int \left(\frac{d\rho}{dt} - \nabla^2\rho^m\right) q,
\end{aligned}$$

where to obtain the last equality we have integrated by parts (twice). Since q is arbitrary, we have recovered the porous medium equation.

Notice that neither $T_\rho\mathcal{M}$ nor g_ρ depend on ρ. Recalling the identification $-\nabla^2 q = s$, we see that formally $\nabla^{-1}s = -\nabla q$, so that

$$g_\rho(s,s) = \int sq = \int (-\nabla^2 q)q = \int |\nabla q|^2 = \int |\nabla^{-1}s|^2,$$

and hence is the H^{-1} norm.

3.2 New Approach

For our new approach, we will instead identify all functions p via

$$-\nabla\cdot(\rho\nabla p) = s,$$

and use the metric tensor

$$g_\rho(s_1,s_2) = \int \rho\nabla p_1\cdot\nabla p_2.$$

Again g_ρ is well–defined and we still have $g_\rho = \int s_1 p_2 = \int s_2 p_1$: we have that

$$\begin{aligned}\nabla \cdot (\rho \cdot p_1 \nabla p_2) &= \nabla(\rho \cdot p_1) \cdot \nabla p_2 + \rho \cdot p_1 \triangle p_2 \\ &= p_1(\nabla \rho \cdot \nabla p_2) + \rho \nabla p_1 \cdot \nabla p_2 + \rho \cdot p_1 \triangle p_2 \\ &= p_1[\nabla \cdot (\rho \nabla p_2)] + \rho \nabla p_1 \cdot \nabla p_2 \\ &= -p_1 s_2 + \rho \nabla p_1 \cdot \nabla p_2.\end{aligned}$$

Since $\rho \to 0$ at ∞, the integral of the left hand size is *zero* by the Divergence Theorem and the result follows (note also that the role of p_1 and p_2 can clearly be interchanged in the above computation).

The metric tensor g_ρ now does indeed depend on ρ and the energy functional is now

$$E(\rho) = \begin{cases} \frac{1}{m-1} \int \rho^m & \text{for } m \neq 1, \\ \int \rho \log \rho & \text{for } m = 1. \end{cases}$$

A similar calculation as in the traditional case shows that we also recover the porous medium equation here.

More motivation will be given for the new approach, but for now let us note that the new identification of q has the advantage that it is blatantly clear that $\int \nabla \cdot (\rho \cdot p_1 \nabla p_2) = 0$, by the divergence theorem, since $\rho \to 0$ at ∞, whereas in the traditional approach, we had to implicitly assume that $\int \nabla \cdot (q_1 \nabla q_2) = 0$, so the new approach seems to allow a wider class of test functions.

4 Physical Derivation of the Porous Medium Equation

To further motivate the new interpretation, and also to familiarize ourselves with some of the variables that will appear, we give a quick physical derivation of the porous medium equation. The derivation is based on three assumptions:

- $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$ (Continuity equation, expressing conservation of mass; ρ denotes the mass density of the gas and u denotes (average) velocity)

- $u = -M \nabla p$ (Darcy's law: $p =$ pressure, and M depends on permeability of medium and viscosity of gas; we will take $M = Id$)

- $p = \frac{\delta E}{\delta \rho}$ (Here E is free energy, and $\frac{\delta E}{\delta \rho}$ denotes the functional derivative, that is

$$\left\langle \frac{\delta E}{\delta \rho}, s \right\rangle = \frac{d}{d\varepsilon} E(\rho + \varepsilon s)|_{\varepsilon=0},$$

for all test functions s. Notice that $\delta E / \delta \rho$ is *not* the same as $\operatorname{diff} E(\rho).s$, which is actually equal to $\langle \frac{\delta E}{\delta \rho}, s \rangle$.

For a free energy of the form $E = \int e(\rho)$, we see that

$$p = e'(\rho) \quad (= \delta E / \delta \rho).$$

Since Darcy's law allows us to eliminate u in favor of p, we see that if we plug everything in, the continuity equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla (e'(\rho))) \equiv \frac{\partial \rho}{\partial t} - \nabla^2 \pi(\rho) = 0,$$

so that

$$\pi(\varphi) = \varphi e'(\varphi) - e(\varphi).$$

We see that to obtain the porous medium equation, we need

$$\pi(\varphi) = \varphi^m,$$

i.e.

$$e(\varphi) = \begin{cases} \frac{1}{m-1}\varphi^m & \text{for } m \neq 1 \\ \varphi \log \varphi & \text{for m} = 1, \end{cases}$$

in accordance with the definition of E in our new interpretation of the porous medium equation as a gradient flow. So in the new interpretation, E is genuinely the free energy.

Finally, let us take a look at the metric tensor. Recall that by definition we have

$$g_\rho(s,s) = \int \rho |\nabla p|^2, \quad \text{where} \quad -\nabla \cdot (\rho \nabla p) = s.$$

Notice that this can be reformulated as

$$g_\rho(s,s) = \inf_u \left\{ \int \rho |u|^2 : s + \nabla \cdot (\rho u) = 0 \right\}.$$

Indeed, we first observe that the solution to the minimization problem has the form $u = -\nabla p$ for some p. To see this, first notice that any u can be written uniquely as an orthogonal sum of a divergence free part and a part which is the gradient of some $\varphi \in C_0^\infty$:

$$u = \psi + \nabla \varphi, \quad \nabla \cdot \psi = 0.$$

Indeed, a simple integration by parts shows that a divergence free function is always perpendicular to a gradient function; on the other hand, if $\varphi = \triangle^{-1}(\nabla \cdot u)$, then $\nabla \cdot (u - \nabla \varphi) = \nabla \cdot (u - \nabla(\triangle^{-1}(\nabla \cdot u))) = 0$, so $u - \nabla \varphi$ is divergence free and hence we can explicitly write

$$u = (u - \nabla \varphi) + \nabla(\triangle^{-1}(\nabla \cdot u)).$$

This fact is enough since to satisfy the continuity equation in the constraint, the divergence free part does not contribute anything and therefore the u with minimal norm would have zero divergence free part. Hence if u solves the minimization problem, then $u = -\nabla p$ for some p and further by the constraint $s = -\nabla \cdot (\rho \nabla p)$ we get

$$\int \rho |u|^2 = \int \rho \nabla p \cdot \nabla p = g_\rho(s,s).$$

Finally we remark that the integral $\int \rho|u|^2$ corresponds to the dissipation of kinetic energy as the gas moves with velocity u. Hence the gradient flow equation $\frac{d}{dt}E(\rho) = -g_\rho\left(\frac{d\rho}{dt}, \frac{d\rho}{dt}\right)$ says exactly that the rate of change of the free energy is given by the rate of dissipation of kinetic energy, hence the energetics defines the functional E while the kinetics define the metric tensor g.

5 Total Derivatives, Covariant Derivatives, and Submersions

The Riemannian metric g induces a distance on $\mathcal{M}$, i.e. given $\rho_0, \rho_1 \in \mathcal{M}$, consider the space of all smooth curves $\sigma \to \rho(\sigma) \in \mathcal{M}$ connecting ρ_0 and ρ_1 and define the distance to be the infimum of the (kinetic) energy over all such curves:

$$d(\rho_0, \rho_1)^2 = \inf\left\{\int_0^1 g\left(\frac{d\rho}{d\sigma}, \frac{d\rho}{d\sigma}\right) : \sigma \to \rho(\sigma) \in \mathcal{M}, \rho(0) = \rho_0, \rho(1) = \rho_1\right\}.$$

We would now like to identify this induced distance on $\mathcal{M}$. We will do so by obtaining the metric g as the "push–forward" of a *flat* metric (i.e. a metric which does not vary from point to point) g^*, defined on a bigger space $\mathcal{M}^*$.

First, we introduce the analogue of the Jacobian. Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between them. Then F induces a linear map

$$DF : T_p\mathcal{M} \to T_{F(p)}\mathcal{N},$$

such that if $f \in C^\infty(\mathcal{N})$, then

$$[DF(X)]f = d(f \circ F)(X),$$

that is, the directional derivative of f in the direction of $DF(X) \in T_{F(p)}(\mathcal{M})$ is the same as the directional derivative of the pull–back of f in the direction of $X \in T_p(\mathcal{M})$, for all $f \in C^\infty(\mathcal{N})$. It can be seen that in Euclidean space DF is exactly given by the Jacobian matrix: Consider $F : \mathbf{R}^n \to \mathbf{R}^m$ and let $f : \mathbf{R}^m \to \mathbf{R}$, then it can be checked explicitly that

$$df(DF(\mathbf{x})) = \nabla f \cdot J\mathbf{x} = \nabla(f \circ F) \cdot \mathbf{x} = d(f \circ F)(\mathbf{x}),$$

where J is the Jacobian matrix and $\mathbf{x} \in \mathbf{R}^n$. Let's observe in particular that if we have a curve $\gamma(t)$ in $\mathcal{M}$ such that $\gamma(0) = p$, then

$$DF(\gamma'(0)) = \frac{d}{dt}F(\gamma(t))|_{t=0}.$$

(This can be seen e.g. by taking smooth coordinate charts seeing that DF is still represented by the Jacobian matrix (see [2], pages 69-72). We can then finish by Taylor expanding both sides.)

Let us now explain how to take the derivative of one vector field in the direction of another vector field. This is the notion of covariant derivative (see [4], Section 1.1.2): In Euclidean space, if X and Y are vector fields, then a natural way to define $\nabla_Y X$ is

$$\nabla_Y X = \langle d(X_i)(Y), \ldots, d(X_N)(Y), \rangle$$

that is, we measure the change in X by measuring how the coefficients change, in the direction given by Y. This is unfortunately not invariantly defined, but it can be checked that this can be defined implicitly as

$$2g(\nabla_Y X, Z) = (L_X g)(Y, Z) + (d\theta_X)(Y, Z),$$

where $\theta_X(Y) = g(X, Y)$ is a 1–form and L_X denotes the Lie derivative. For a general Riemannian manifold we will have to use this implicit definition, and there are four defining properties of ∇:

- (Tensoriality) $\nabla_{\alpha v+\beta w} X = \alpha \nabla_v X + \beta \nabla_w X$.
- (Derivation) $\nabla_Y (X_1) + X_2) = \nabla_Y X_1 + \nabla_Y X_2$ and $\nabla_Y (fX) = (D_Y f)X + f\nabla_Y X$.
- (Torsion Free) $\nabla_X Y - \nabla_Y X = [X, Y]$.
- (Metric) $D_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

In particular, we note an "application" of the last property. Suppose $p : \mathbf{R}^N \to \mathbf{R}$ is a scalar function. We may then use the metric property to compute, with $u = \nabla p$ and ∂_i denoting the i^{th} coordinate direction (∇_{∂_i} denotes either the directional derivative or the covariant derivative, depending on the context)

$$(\nabla|\nabla p|^2)_i = \nabla_{\partial_i}(|\nabla p|^2) = \nabla_{\partial_i}(\nabla p, \nabla p) = \nabla_{\partial_i}(u, u) = 2(\nabla_{\partial_i} u, u) = 2[Du.u]_i,$$

where Du denotes the Jacobian matrix of u.

Finally we need the notion of a Riemannian submersion (see [4], page 4). Let $F : \mathcal{N} \to \mathcal{M}$ be a smooth map such that DF is surjective at each point. Hence, as vector spaces,

$$(\ker DF)^{\perp} \cong T_{F(\rho)}(\mathcal{M})$$

for each ρ, and in a natural way, $g_{\mathcal{N}}$ can be pushed forward to be a metric on $\mathcal{M}$:

$$g_{\mathcal{M}}(DF(v), DF(w)) = g_{\mathcal{N}}(v, w),$$

for each $v, w \in (\ker DF)^{\perp}$. Conversely, suppose we are given a smooth map $F : \mathcal{N} \to \mathcal{M}$, and we wish to show it is a submersion. It is enough to show that DF restricted $(\ker DF)^{\perp}$ is an isometry (for the more general statement $g_{\mathcal{M}}(F(v), F(w)) = g_{\mathcal{N}}(v, w)$, consider $g(v-w, v-w)$). We claim this is equivalent to showing

$$g_{\mathcal{M}}(s, s) = \inf_{DF(v)=s} g_{\mathcal{N}}(v, v).$$

Indeed, this follows from the facts that we have in the usual way that 1) any element $v \in \mathcal{N}$ can be written as $v = v_0 + v^{\perp}$, where $v_0 \in \ker DF, v^{\perp} \in (\ker DF)^{\perp}$, so that $g(v, v) = g(v_0, v_0) + g(v^{\perp}, v^{\perp})$, and 2) DF is injective when restricted to $(\ker DF)^{\perp}$.

6 Submersion Into the Lagrangian Description

So far we have described the evolution via the particle density ρ (an Eulerian description). We may alternatively describe the situation via coordinates of the particle, i.e. the flow map Φ (a Lagrangian description). We will attempt to understand the geometry of $\mathcal{M}$ via the *flat* geometry obtained from the Lagrangian description.

Before giving formal definitions, let us first try to understand how the evolution of the particle density is given by some flow map. Recall that 1) given a vector field v, we may define the flow Φ by the differential equation $\dot{\Phi} = v$, $\Phi(0) = \mathrm{Id}$ and 2) the continuity equation which says that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

relates the particle density to the velocity, given via a vector field u; we note that implicitly u depends on time. With a single particle, we can push it around under Φ and therefore it is natural to do the same when we have a distribution of particles, which leads to the notion of the push forward of a density. We write

$$\rho = \Phi \# \rho_0$$

if for all functions ζ, we have

$$\int \zeta \rho = \int (\zeta \circ \Phi) \rho_0.$$

Now we note that if we define $\rho_t = \Phi_t \# \rho_0$, where Φ_t is the flow corresponding to u, then the continuity equation is satisfied. Indeed, let $\zeta \in C_c(\mathbf{R}^N)$, then

$$\begin{aligned}
\frac{d}{dt} \int \zeta \rho &= \int \frac{d}{dt} (\zeta \circ \Phi_t) \rho_0 \\
&= \int (\nabla \zeta(\Phi_t) \cdot u_t(\Phi_t)) \rho_0 \\
&= \int (\nabla \zeta \cdot u_t) \rho_t \\
&= - \int \zeta \nabla \cdot (\rho_t u_t),
\end{aligned}$$

and so ρ_t solves the continuity equations.

Let us then consider the manifold

$$\mathcal{M}^* = \{\text{diffeomorphisms } \Phi : \mathbf{R}^N \to \mathbf{R}^N\}$$

and the map

$$\Pi : \mathcal{M}^* \to \mathcal{M} : \Phi \to \Phi \# \rho_0,$$

where ρ_0 is some fixed density, which we envision as the initial density. We envision the tangent space as

$$T_\Phi \mathcal{M}^* = \{\text{vector fields } v : \mathbf{R}^N \to \mathbf{R}^N\},$$

with the flat metric tensor
$$g^*_\Phi(v_1,v_2)=\int (v_1\cdot v_2)\rho_0.$$
We now claim that $D\Pi_\Phi : T_\Phi\mathcal{M}^* \to T_\rho\mathcal{M}$ is a Riemannian submersion. Recall that this means we have to show
$$g_\rho(s,s)=\inf_{D\Pi_\Phi(v)=s} g^*_\Phi(v,v),$$
for all $s\in T_\rho\mathcal{M}$ (again recall $g_\rho(s,s)=\int\rho|\nabla p|^2$, where $\nabla\cdot(\rho\nabla p)=s$).

To this end we first make further identification on $T_\Phi\mathcal{M}^*$ via
$$v=u\circ\Phi.$$
(That is, we identify vector fields u_1 and u_2 if $u_1\circ\Phi=u_2\circ\Phi$.) Notice that with this identification, the metric tensor becomes
$$g^*_\Phi(v_1,v_2)=\int(v_1\cdot v_2)\rho_0=\int(u_1\cdot u_2)\rho,$$
where $\rho=\Pi(\Phi)$. Now let us try to characterize $D\Pi_\Phi$, $\ker D\Pi_\Phi$, and $(\ker D\Pi_\Phi)^\perp$. First let us figure out what is $D\Pi_\Phi$. From our discussion about the continuity equation, we see that we should have $\frac{d\rho}{dt}=-\nabla\cdot(\rho u)$; on the other hand, our identification in $T_\rho\mathcal{M}$ says that $\nabla\cdot(\rho\nabla p)=s$. It is then not surprising that $D\Pi_\Phi(u)$ should be the function p on $\mathbf{R}^N$ satisfying
$$-\nabla\cdot(\rho\nabla p)=-\nabla\cdot(\rho u).$$
Let us assume this for now and show a few things:

- ($\ker D\Pi_\Phi$) This characterization of $D\Pi_\Phi(u)$ immediately implies that $u\in\ker D\Pi_\Phi$ if and only if
$$\nabla\cdot(\rho u)=0.$$
(Indeed, if $u\in\ker D\Pi_\Phi$, then clearly $\nabla\cdot(\rho u)=0$; conversely, if $\nabla\cdot(\rho u)=0$, then $-\nabla\cdot(\rho\nabla p)=0$ and hence p is *identified* with $s=0$.)

- ($(\ker D\Pi_\Phi)^\perp$) Next we claim that $w\in(\ker D\Pi_\Phi)^\perp$ if and only if there exists some function p such that
$$w=\nabla p.$$
Indeed first suppose $w\in(\ker D\Pi_\Phi)^\perp$ and let us write $w=w_0+\nabla\varphi$ as an orthogonal sum, where $\nabla\cdot w_0=0$ and $\varphi\in C_c(\mathbf{R}^N)$. By definition of g^*_Φ and our characterization of $\ker D\Pi_\Phi$, this means that $\int(w\cdot u)\rho=0$ for all u such that $\nabla\cdot(\rho u)=0$. That is, for all such u, we have
$$\int w_0\cdot(\rho u)=\int(w_0\cdot u)\rho=-\int(\nabla\varphi\cdot u)\rho=\int\nabla\cdot(\varphi\rho u)=0,$$
so $w_0=0$: This shows that w_0 is perpendicular to any divergence free vector field, in addition to all gradients, and hence we conclude $w=\nabla\varphi$. Conversely, if $w=\nabla\varphi$, then clearly the same integration by parts as above shows that $\int(w\cdot u)\rho=0$ for any $u\in\ker D\Pi_\Phi$.

- (Riemannian Submersion) Let us now write $-\nabla\cdot(\rho\nabla p) = -\nabla\cdot(\rho u)$ in variational form as

$$\int (\nabla p\cdot\nabla\zeta)\rho = \int (u\cdot\nabla\zeta)\rho,$$

for all $\zeta : \mathbf{R}^N \to \mathbf{R}$. If we establish this, then we obtain $\int|\nabla p|^2\rho = \int(u\cdot\nabla p)\rho \le \int |u|^2\rho$ and hence by our characterization of $(\ker D\Pi_\Phi)^\perp$, we have

$$\int |\nabla p|^2\rho = \inf_{D\Pi_\Phi(u)=p}\int |u|^2\rho.$$

(In particular, the infimum is achieved by u such that $u = \nabla p$, which we showed satisfies $D\Pi_\Phi(u) = p$.) Recalling the definition of g_ρ and g^*_Φ, we see that we have established that Π is indeed a Riemannian submersion.

It remains to establish our (variational) characterization of $D\Pi_\Phi(u)$. Consider the curve in $\mathcal{M}^*$ given by

$$\frac{\partial\tilde\Phi}{\partial\sigma}(\sigma) = u\circ\tilde\Phi(\sigma), \quad \tilde\Phi(0) = \Phi.$$

This curve clearly goes through Φ at $\sigma = 0$, and we may map it under Π to obtain a curve $\sigma\to\tilde\rho(\sigma)$ in $\mathcal{M}$ which goes through ρ at $\sigma = 0$. It is enough to show that the tangent to this image curve at $\sigma = 0$ satisfies the variational formulation. On the one hand, by definition of Π,

$$\int \tilde\rho(\sigma)\zeta = \int (\zeta\circ\tilde\Phi(\sigma))\rho_0,$$

so that

$$\int \frac{\partial\tilde\rho}{\partial\sigma}_{|\sigma=0}\zeta = \int [(\nabla\zeta\circ\Phi)\cdot(u\circ\Phi)]\rho_0 = \int (\nabla\zeta\cdot u)\rho.$$

On the other hand, denoting $\frac{\partial\tilde\rho}{\partial\sigma}_{|\sigma=0} = s$ and recalling the identification $-\nabla\cdot(\rho\nabla p) = s$, we see that

$$\int s\zeta = -\int \nabla\cdot(\rho\nabla p)\zeta = \int (\nabla\zeta\cdot\nabla p)\rho.$$

7 Geodesics and a Property of Π

We now derive some more properties of Π. First we need the notion of a geodesic. Informally, a geodesic is a curve with constant speed and is described by the differential equation $\ddot\gamma = 0$. E.g. in Euclidean space, if $\gamma : [0,1]\to\mathbf{R}^N$ is a curve, then $\ddot\gamma = (\ddot\gamma_1(t),\ldots,\ddot\gamma_N(t))$, and we quickly conclude that geodesics are straight lines, i.e. $\gamma(t) = \gamma(0) + vt$ where $v = (\dot\gamma_1(0),\ldots,\dot\gamma_N(0))$.

This concept in general depends on the metric as can be seen if we attempt to take a second derivative of some curve $\gamma(s,t)$ (for convenience we assume γ takes two real parameters):

$$\frac{\partial^2\gamma}{\partial s\partial t} = \frac{\partial}{\partial s}(\frac{\partial\gamma^i}{\partial t}\partial_i) = \frac{\partial^2\gamma^i}{\partial s\partial t}\partial_i + \frac{\partial\gamma_i}{\partial t}(\frac{\partial}{\partial s}\partial_i) = \frac{\partial^2\gamma^i}{\partial s\partial t}\partial_i + \frac{\partial\gamma_i}{\partial t}\nabla_{\frac{\partial\gamma}{\partial s}}\partial_i,$$

and the covariant derivative certainly depends on the metric g. However, we will build all this into our definition of second derivatives and interpret

$$\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}.$$

Further, we note that we therefore have that the metric property is satisfied:

$$\frac{\partial}{\partial t^k} g\left(\frac{\partial \gamma}{\partial t^i}, \frac{\partial \gamma}{\partial t^j}\right) = g\left(\frac{\partial^2 \gamma}{\partial t^k \partial t^i}, \frac{\partial \gamma}{\partial t^j}\right) + g\left(\frac{\partial \gamma}{\partial t^i}, \frac{\partial^2 \gamma}{\partial t^k \partial t^j}\right).$$

We will assume that given an initial point p and initial direction (some $v \in T_p\mathcal{M}$), geodesics exist and are unique (for more on this see [4], Section 5.2).

Recall that our eventual goal is to identify the distance on $\mathcal{M}$ from the distance on $\mathcal{M}^*$. The first step is identification of the geodesics, as geodesics define the distance. Since geodesics depend only on the metric, it is indeed the case that an isometry takes geodesics to geodesics. In the case of a submersion, we have to be a bit more careful: We need to ensure that tangents of the geodesics remain in $(\ker T\Pi_\Phi)^\perp$. In our case, this is the property that if $\sigma \mapsto \Phi(\sigma)$ is a geodesic in $\mathcal{M}^*$, then

$$\frac{d\Phi}{d\sigma}(0) \in (\ker D\Pi_{\Phi(0)})^\perp \text{ implies } \frac{d\Phi}{d\sigma}(\sigma) \in (\ker D\Pi_{\Phi(\sigma)})^\perp \text{ for all } \sigma.$$

We will now establish this via a uniqueness argument involving a Hamilton–Jacobi equation.

The geodesic equation in $\mathcal{M}^*$ says $\frac{d^2\Phi}{d\sigma^2} = 0$. Let us write $\frac{d\Phi}{d\sigma} = u \circ \Phi$, where $u \mapsto u(\sigma, \Phi)$ is the tangent field. Then we see that

$$\begin{aligned}
\frac{d^2\Phi}{d\sigma^2} &= \frac{d}{d\sigma}(u \circ \Phi) \\
&= \frac{\partial u}{\partial \sigma} \circ \Phi + \langle \frac{\partial u_1}{\partial x_i}\frac{\partial \Phi_i}{\partial \sigma}, \dots, \frac{\partial u_N}{\partial x_i}\frac{\partial \Phi_i}{\partial \sigma}\rangle \\
&= \frac{\partial u}{\partial \sigma} \circ \Phi + (Du \circ \Phi) \cdot \frac{d\Phi}{d\sigma} \\
&= \left(\frac{\partial u}{\partial \sigma} + Du.u\right) \circ \Phi,
\end{aligned}$$

where D denotes the Jacobian matrix in the spatial variables and the index i is to be summed over in the second to last line. The geodesic equation then becomes

$$\frac{\partial u}{\partial \sigma} + Du.u = 0.$$

Now if $\sigma \mapsto \Phi(\sigma)$ is a geodesic in $\mathcal{M}^*$ and $\frac{d\Phi}{d\sigma}(0) \in (\ker D\Pi_{\Phi(0)})^\perp$, then by our previous characterization, there exists some p_0 such that

$$\nabla p_0 = \frac{d\Phi}{d\sigma}(0) = u(0)$$

and therefore the tangent field u associated to this geodesic satisfies the displayed PDE with $u(0) = \nabla p_0$.

On the other hand, we can let $\tilde{p}(\sigma, x)$ solve the Hamilton–Jacobi equation

$$\frac{\partial \tilde{p}}{\partial \sigma} + \frac{1}{2}|\nabla \tilde{p}|^2 = 0$$

with initial data p_0. Now if $\tilde{u} = \nabla \tilde{p}$, then a direct computation shows

$$\frac{\partial \tilde{u}}{\partial \sigma} = -\frac{1}{2}\nabla|\nabla \tilde{p}|^2 = -D\tilde{u}.\tilde{u}.$$

We conclude that u and $\tilde{u}$ solve the same equation with identical initial date, so $\tilde{u} = u$ and hence

$$u(\sigma) = \nabla \tilde{p}(\sigma),$$

for all σ. But this by our previous characterization means exactly that $\frac{d\Phi}{d\sigma}(\sigma) = u(\sigma) \in (\ker D\Pi_{\Phi(\sigma)})^{\perp}$, for all σ.

8 More Properties of the Geodesic under Π

We will need a few observations about the behavior of geodesics under Π before we can identify the geodesics in $\mathcal{M}$ from those in $\mathcal{M}^*$.

- (**Energy is Descreasing Under** Π) First note how energy transforms under Π: Suppose $\sigma \mapsto \Phi(\sigma)$ is a curve in $\mathcal{M}^*$ and consider its image $\sigma \mapsto \rho(\sigma)$ in $\mathcal{M}$ under Π. Since we have $\rho(\sigma) = \Pi(\Phi(\sigma))$, we have
$$\frac{d\rho}{d\sigma} = D\Pi_{\Phi(\sigma)}\frac{d\Phi}{d\sigma}.$$
Since Π is a submersion and we have $g_\rho(s,s) = \inf_{D\Pi_\Phi.v=s} g^*_\Phi(v,v)$, we clearly have
$$\int g_\rho\left(\frac{d\rho}{d\sigma}, \frac{d\rho}{d\sigma}\right)\, d\sigma \leq \int g^*_\Phi\left(\frac{d\Phi}{d\sigma}, \frac{d\Phi}{d\sigma}\right)\, d\sigma,$$
with equality if (and only if) $\frac{d\Phi}{d\sigma} \in (\ker D\Pi_\Phi)^{\perp}$.

- (**Images of Geodesics Are Geodesics**) Next we observe that basically, images of geodesics are geodesics. This is fairly clear since we have that $T_\rho\mathcal{M} \cong (\ker D\Pi_\Phi)^{\perp}$ (here $\rho = \Pi(\Phi)$), and geodesics only depend on the metric g and the property in the previous section guarantees exactly that geodesics remain in $(\ker D\Pi_\Phi)^{\perp}$ if it starts out there.

  More precisely, recall for us the induced distance is defined as
$$d(\rho_0,\rho_1)^2 = \inf\left\{\int_0^1 g\left(\frac{d\rho}{d\sigma}, \frac{d\rho}{d\sigma}\right) : \sigma \to \rho(\sigma) \in \mathcal{M}, \rho(0) = \rho_0, \rho(1) = \rho_1\right\}.$$
  By studying variations of this energy, we can show that minima of this energy must be geodesics: Given a curve γ (which we assume to be smooth), let's denote
$$E(\gamma) = \int_0^1 g(\dot{\gamma}, \dot{\gamma})\, dt.$$

We can then consider proper variations of γ, i.e. some smooth $\gamma(s,t)$ such that $\gamma(0,t)=\gamma(t)$ and both endpoints are fixed for all s, so in particular $\frac{\partial\gamma}{\partial s}|_{s=0}=0$. If γ minimizes E, then

$$\frac{dE}{ds}|_{s=0}=0,$$

but then by the metric property of second derivatives

$$\begin{aligned}\frac{d}{ds}E(\gamma)|_{s=0} &= 2\int_0^1 g\left(\frac{\partial^2\gamma}{\partial s\partial t},\frac{\partial\gamma}{\partial t}\right)\,dt\\ &= 2\int_0^1 \frac{\partial}{\partial t}g\left(\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t}\right)-g\left(\frac{\partial\gamma}{\partial s},\frac{\partial^2\gamma}{\partial t^2}\right)\,dt\\ &= -2\int_0^1 g(V(t),\ddot\gamma)\,dt,\end{aligned}$$

where $V(t)=\frac{\partial\gamma}{\partial s}|_{(0,t)}$ denotes the variational field. Any such variational field with $V(0)=V(1)=0$ gives a variation of γ and hence in this case if we choose $V(t)=\ddot\gamma$, then we see that

$$0=\frac{dE}{ds}|_{s=0}=-\int g(\ddot\gamma,\ddot\gamma)\,dt,$$

and hence $\ddot\gamma=0$ and γ is a geodesic. We have done this for the case where γ is smooth, but another choice of $V(t)$ actually yields the result for piecewise smooth curves (see [4], Section 5.4 pages 127-129, especially Theorem 13). We note that in general the converse is not true: For example, on the sphere geodesics are part of great circles, and these clearly don't always minimize distance.

So now suppose $\sigma\mapsto\Phi(\sigma)$ is a geodesic on $(\mathcal{M}^*,g^*)$ with $\frac{d\Phi}{d\sigma}\in(\ker D\Pi_\Phi)^\perp$, then $\sigma\mapsto\rho(\sigma)$ is a geodesic on $(\mathcal{M},g)$: It is enough to show that it is an energy minimizing curve, so if $(\varepsilon,\sigma)\mapsto\tilde\rho(\varepsilon,\sigma)$ is a variation of $\sigma\mapsto\rho(\sigma)$, then we may lift it to be a variation $(\varepsilon,\sigma)\mapsto\tilde\Phi(\varepsilon,\sigma)$ (such that $\tilde\Phi(0,\sigma)=\Phi(\sigma)$) with $\frac{d\tilde\Phi}{d\sigma}\in(\ker T_{\tilde\Phi}\Pi)^\perp$, and hence

$$\int g_\rho\left(\frac{d\rho}{d\sigma},\frac{d\rho}{d\sigma}\right)\,d\sigma=\int g^*_\Phi\left(\frac{d\Phi}{d\sigma},\frac{d\Phi}{d\sigma}\right)\,d\sigma\le\int g^*_{\tilde\Phi}\left(\frac{d\tilde\Phi}{d\sigma},\frac{d\tilde\Phi}{d\sigma}\right)\,d\sigma=\int\left(\frac{d\tilde\rho}{d\sigma},\frac{d\tilde\rho}{d\sigma}\right)\,d\sigma.$$

- (**Onto: Geodesics Come From Geodesics**) Conversely, if $\sigma\mapsto\rho(\sigma)$ is a geodesic with $\rho(0)=\rho_0$, then there exists a geodesic $\sigma\mapsto\Phi(\sigma)$ with $\Phi(0)=$id, $\frac{d\Phi}{d\sigma}\in(\ker D\Pi_\Phi)^\perp$ such that $\sigma\mapsto\rho(\sigma)$ is its image under Π: This is clear as we may simply consider the geodesic in $\mathcal{M}^*$ with the prescribed boundary condition and such that $\frac{d\Phi}{d\sigma}(0)\in(\ker D\Pi_{\Phi(0)})^\perp$. The property we established in the previous section implies the belonging to kernel perp property is preserved along the geodesic. Thus, by the previous observation, the image of this geodesic under Π is a geodesic in $\mathcal{M}$ satisfying the same initial data as $\sigma\mapsto\rho(\sigma)$ and hence must coincide with it.

Finally, combining the previous observations, we see that in fact

$$d(\rho_0, \rho)^2 = \inf_{\Pi(\Phi)=\rho} d^*(\mathrm{id}, \Phi)^2.$$

Indeed given any Φ, consider any curve $\sigma \mapsto \tilde{\Phi}(\sigma)$ such that $\tilde{\Phi}(0) = \mathrm{id}$ and $\tilde{\Phi}(1) = \Phi$, then $d(\rho_0, \rho)^2 \leq d^*(\mathrm{id}, \Phi)^2$ comes from the definition of d and the fact that energy is decreasing under Φ. On the other hand, we may take an energy minimizing curve $\sigma \mapsto \tilde{\rho}(\sigma)$ connecting ρ_0 to ρ and use the "onto" property of geodesics under Π and the definition of d^* to produce some Φ such that the opposite inequality holds.

9 Identification of Geodesics and the Induced Distance

Now it is fairly straightforward to identify the geodesics and induced distance in $(\mathcal{M}, g)$ from those in $(\mathcal{M}^*, g^*)$. Now let $\sigma \mapsto \rho(\sigma)$ be a geodesic with initial data

$$\rho(0) = \rho_0 \quad \text{and} \quad \frac{d\rho}{d\sigma}(0) = s$$

and consider the corresponding geodesic in $(\mathcal{M}^*, g^*)$ with

$$\Phi(0) = \mathrm{id}, \quad D\Pi_{\mathrm{id}}\left(\frac{d\Phi}{d\sigma}(0)\right) = s \quad \text{and} \quad \frac{d\Phi}{d\sigma} \in (\ker D\Pi_{\Phi})^{\perp}, \text{for all } \sigma$$

and

$$\rho(\sigma) = \Phi(\sigma)\#\rho_0, \quad \text{for all } \sigma.$$

By our representation of the tangent vectors and our characterization of $D\Pi_{\mathrm{id}}$, we have that

$$-\nabla \cdot (\rho_0 \nabla p) = s = D\Pi_{\mathrm{id}}\left(\frac{d\Phi}{d\sigma}(0)\right) = -\nabla \cdot \left(\rho_0 \frac{d\Phi}{d\sigma}(0)\right).$$

Since $\frac{d\Phi}{d\sigma}(0) \in (\ker D\Pi_{\mathrm{id}})^{\perp}$, we have $\frac{d\Phi}{d\sigma}(0) = \nabla \tilde{p}$, for some $\tilde{p}$, which by the previous display we may take to be p (since we identify p's which solve the same equation $\nabla \cdot (\rho_0 \nabla p) = s$). Recall the geodesic equation in $(\mathcal{M}^*, g^*)$ is given by

$$\frac{\partial^2 \Phi}{\partial \sigma^2} = 0.$$

With initial conditions $\Phi(0) = \mathrm{id}$ and $\frac{d\Phi}{d\sigma}(0) = \nabla p$, we see that we have Φ of the form (note that $\frac{1}{2}\nabla|y|^2 = y$)

$$\Phi(\sigma) = \nabla\left(\frac{1}{2}|y|^2 + \sigma p\right).$$

Therefore the geodesics in $(\mathcal{M}, g)$ are given by

$$\rho(\sigma) = \left[\nabla\left(\frac{1}{2}|y|^2 + \sigma p\right)\right]\#\rho_0,$$

where

$$\rho(0) = \rho_0 \quad \text{and} \quad \frac{d\rho}{d\sigma}(0) = -\nabla \cdot (\rho_0 \nabla p).$$

Finally we identify the induced distance. We have

$$d(\rho_0, \rho)^2 = \inf_{\rho = \Phi \# \rho_0} d^*(\mathrm{id}, \Phi)^2$$

Now recall that $\mathcal{M}^*$ is flat and hence the geodesic equaion really *is* $\ddot{\Phi} = 0$, so in particular given Φ_1, Φ_2,

$$t\Phi_1 + (1-t)\Phi_2$$

is a geodesic. Since the curvature is zero for $\mathcal{M}^*$, by the second variational formula for energy (see [4], Section 6.2, Theorem 21), geodesics indeed do minimize energy (which we take to be the distance squared here as well), and hence (recall $g^*_\Phi(v_1, v_2) = \int \rho_0 v_1 \cdot v_2$)

$$d^*(\Phi_1, \Phi_2) = \int_0^1 \int \rho_0 |\Phi_1 - \Phi_2|^2 \, dt = \int \rho_0 |\Phi_1 - \Phi_2|^2.$$

So we finally conclude

$$d(\rho_0, \rho)^2 = \inf_{\rho = \Phi \# \rho_0} \int \rho_0 |\mathrm{id} - \Phi|^2.$$

10 The Wasserstein Metric

The induced distance from the previous section is basically the Wasserstein metric. Indeed, we may imagine a mass transference problem from a density ρ_0 to ρ_1 minimizing over the cost function given by the Euclidean distance squared. The existences of Φ such that $\rho_1 = \Phi \# \rho_0$ corresponds to the existence of a solution to the so–called Monge's problem.

In general, we ask for a relaxed version of the problem by allowing mass to be "split". More precisely, let μ_0 and μ_1 be nonnegative Borel probability measures and consider the set of couplings of μ_0 and μ_1:

$$\begin{aligned} P(\mu_0, \mu_1) = \{&\text{Borel probability measures } \mu \text{ on } \mathbf{R}^N \times \mathbf{R}^N \mid \\ &\int \zeta(y_0)\, \mu(dy_0, dy_1) = \int \zeta(y_0)\, \mu_0(dy_0) \text{ and} \\ &\int \zeta(y_1)\, \mu(dy_0, dy_1) = \int \zeta(y_1)\, \mu_1(dy_1) \text{ for all } \zeta \in C_0^\infty(\mathbf{R}^N)\}. \end{aligned}$$

So basically any "question" we ask about the first "coordinate" has an answer which is the same as if we were just using the μ_0 measure and similarly for the second "coordinate". Said differently, we have

$$\pi_1 \# \mu = \mu_0 \text{ and } \pi_2 \# \mu = \mu_1,$$

where π_1 and π_2 corresponds to projection onto the first and second coordinates, respectively, so e.g. if $A \in \mathbf{R}_1^N$, then $\mu_0(A) = \pi_1 \# \mu(A) = \mu(A \times \mathbf{R}_2^N)$. We then define the Wasserstein distance between μ_0 and μ_1 as

$$d(\mu_0, \mu_1) = \inf_{\mu \in P(\mu_0,\mu_1)} \int |y_1 - y_0|^2 \ \mu(dy_0, dy_1).$$

This minimization problem is the Kantorovich problem and the minimizing μ is called a transference plan.

Let us note that the product measure $\mu_0 \times \mu_1$ is always contained in $P(\mu_0, \mu_1)$. Here each "particle" in the support of μ_0 is transported into all of the support of μ_1, with weight given by μ_1. This is in contrast to the situation when Monge's problem has a one–to–one solution: each particle x is transported to $\Phi(x)$ (which is unique). In the case where we have densities ρ_0 and ρ_1 and Φ exists, the corresponding coupling is given by

$$\mu = (\mathrm{id} \times \Phi)\#\rho_0,$$

so then

$$\int \zeta(x_0, x_1) \ \mu(dx_0, dx_1) = \int \zeta(x_0, \Phi(x_0))\rho_0(x_0) \ dx_0,$$

and so

$$\int |\mathrm{id}(x_0) - \Phi(x_0)|^2 \rho_0(x_0) \ dx_0 = \int |x_0 - x_1|^2 \ \mu(dx_0, dx_1).$$

Finally, we point out that in the case $\mu_0 = \rho_0 \ dx_0$ and $\mu_1 = \rho_1 \ dx_1$ have bounded support, it has been shown (by Brenier in 1991) that the Kantorovich problem has a unique solution given by

$$\mu = (\mathrm{id} \times \nabla\varphi)\#\rho_0,$$

where φ is a convex function. So here $\rho_1 = \nabla\varphi\#\rho_0$ and in this case the relaxation is artificial. Recall also in our case we have

$$\varphi_\sigma(y) = \frac{1}{2}|y|^2 + \sigma p(y).$$

11 Explanation and Statement of Results

The porous medium equation admits an exact self–similar solution of the form

$$\rho_*(t, x) = \frac{1}{t^{N\alpha}} \hat{\rho}_* \left(\frac{x}{t^\alpha}\right)$$

where $\hat{\rho}_*$ is given implicitly as

$$e'(\hat{\rho}_*(y)) = \begin{cases} \frac{m}{m-1}\hat{\rho}_*(y)^{m-1} = \max\{\lambda - \alpha\frac{1}{2}|y|^2, 0\} & \text{for } m > 1 \\ \log\hat{\rho}_*(y) + 1 = \lambda - \alpha\frac{1}{2}|y|^2 & \text{for } m = 1 \\ \frac{m}{m-1}\hat{\rho}_*(y)^{m-1} = \lambda - \alpha\frac{1}{2}|y|^2 & \text{for } m < 1 \end{cases},$$

where

$$e(\rho) = \begin{cases} \frac{1}{m-1}\rho^m & \text{for } m \neq 1 \\ \rho\log\rho & \text{for } m = 1 \end{cases},$$

so that $E(\rho) = \int e(\rho)$,

$$\alpha = \frac{1}{N(m-1)+2}$$

and λ is such that

$$\int \hat{\rho}_* = 1.$$

The goal is to show that under the rescaling

$$x = t^\alpha y \quad \text{and} \quad t = e^\tau, \quad \text{so that} \quad \rho(t,x) = \frac{1}{t^{N\alpha}} \hat{\rho} \left(\log t, \frac{x}{t^\alpha} \right),$$

$$\hat{\rho} \to \hat{\rho}_*, \text{ as } \tau \to \infty.$$

More precisely, one can check that $\hat{\rho}$ satisfies

$$\frac{\partial \hat{\rho}}{\partial \tau} - \nabla_y^2 \hat{\rho}^m - \alpha \nabla_y \cdot (\hat{\rho} y) = 0$$

and can be interpreted as the gradient flow

$$\frac{d\hat{\rho}}{d\tau} = -\mathrm{grad}\, F|_{\hat{\rho}},$$

where F is now given by

$$F(\hat{\rho}) = E(\hat{\rho}) + \alpha M(\hat{\rho}),$$

that is, F is E plus the second moment

$$M(\hat{\rho}) = \int \frac{1}{2} |y|^2 \hat{\rho}(y) \; dy.$$

We will show three asymptotic results

$$\frac{d}{d\tau}(e^{2\alpha\tau} |\mathrm{grad}\, F_{|\hat{\rho}}|^2) \leq 0,$$

$$\frac{d}{d\tau}(e^{2\alpha\tau}(F(\hat{\rho}) - F(\hat{\rho}_*))) \leq 0,$$

$$\frac{d}{d\tau}(e^{2\alpha\tau} d(\hat{\rho}, \hat{\rho}_*)^2) \leq 0.$$

Here the distance in the last line will turn out to be the Wasserstein distance.

These goals will be achieved roughly in a three step process: 1) formal manipulations 2) proof in the smooth setting 3) approximation argument.

12 F and grad F

First we show that

$$F(\hat{\rho}) - F(\hat{\rho}_*) \geq 0, \quad \text{for all } \hat{\rho} \in \mathcal{M},$$

so $\hat{\rho}_*$ is a minimizer of F and hence $0 = -\mathrm{grad}\, F_{|\hat{\rho}_*}$. In this section for simplicity of notation, we will denote $\hat{\rho} = \rho$. Let us first define

$$H(\rho_1, \rho_0) = \int [e(\rho_1) - e(\rho_0) - e'(\rho_0)(\rho_1 - \rho_0)],$$

where e is the energy density (so that $E(\rho) = \int e(\rho)$) and in case $m < 1$, we take $H(\rho_1, \rho_0) = \infty$ if ρ_0 vanishes on a set of positive measure. Notice that since e is convex, $H \geq 0$. We now claim that

$$F(\rho) - F(\rho_*) \begin{cases} \geq H(\rho, \rho_*) & \text{for } m > 1 \\ = H(\rho, \rho_*) & \text{for } m \leq 1 \end{cases}.$$

We will do the case $m > 1$ (the cases $m < 1$ and $m = 1$ follow by similar reasoning). For $m > 1$, we have $e(\rho) = \frac{1}{m-1}\rho^m$, so that $e'(\rho) = \frac{m}{m-1}\rho^{m-1}$, and hence

$$H(\rho, \rho_*) = E(\rho) - E(\rho_*) - \int \frac{m}{m-1}\rho_*^{m-1}(\rho - \rho_*).$$

Now recall $F(\rho) = E(\rho) + \int \alpha\frac{1}{2}|y|^2\rho(y)\ dy$, so that the above becomes

$$F(\rho) - F(\rho_*) = H(\rho, \rho_*) + \int \left(\frac{m}{m-1}\rho_*^{m-1} + \alpha\frac{1}{2}|y|^2\right)(\rho - \rho_*).$$

Now for $m > 1$, the definition of ρ_* says that $\frac{m}{m-1}\rho_*^{m-1}(y) = \max\{\lambda - \alpha\frac{1}{2}|y|^2, 0\}$. Now if y is such that $\lambda - \alpha\frac{1}{2}|y|^2 \geq 0$, then

$$F(\rho) - F(\rho_*) = H(\rho, \rho_*) + \int \lambda(\rho - \rho_*) = H(\rho, \rho_*),$$

whereas if y is such that the maximum is achieved at 0, then $\rho_* = 0$ and $\lambda \leq \alpha\frac{1}{2}|y|^2$, and we still get instead $F(\rho) - F(\rho_*) \geq H(\rho, \rho_*)$.

Now we claim that

$$|\mathrm{grad}\, F_{|\rho}|^2 = \int \rho|\nabla p|^2,$$

where $p(y) = e'(\rho(y)) + \alpha\frac{1}{2}|y|^2$ is the pressure corresponding to the "rescaled" energy functional F. Recall that the gradient is defined implicitly via

$$g(\mathrm{grad}\, F, X) = dF(X),$$

for all $X \in T_\rho M$. Now note that since $g(v-w, v-w) \geq 0$ and hence $\frac{1}{2}g(v,v) \geq g(v,w) - \frac{1}{2}g(w,w)$ for any $v, w \in T_\rho M$, we have the inequality (with $v = \mathrm{grad}\, F$ and $s = w \in T_\rho M$ arbitrary)

$$\begin{aligned}\frac{1}{2}g(\mathrm{grad}\, F, \mathrm{grad}\, F) &\geq g(\mathrm{grad}\, F, s) - \frac{1}{2}g(s,s) \\ &= \mathrm{diff}\, F_{|\rho}.s - \frac{1}{2}g(s,s),\end{aligned}$$

and hence

$$\frac{1}{2}g(\operatorname{grad} F, \operatorname{grad} F) = \sup_{s \in T_\rho M} \left\{ \operatorname{diff} F_{|\rho}.s - \frac{1}{2}g(s,s) \right\}.$$

Now since $F = \int e(\rho(y)) + \alpha \frac{1}{2}|y|^2 \, dy$, we have that

$$\operatorname{diff} F_{|\rho}.s = \int ps,$$

with p as defined above. Recalling that $g(s,s) = \int \rho |\nabla q|^2$, where $-\nabla \cdot (\rho \nabla q) = s$, we see that after an integration by parts $(\nabla \cdot (p\rho \nabla q) = p\nabla \cdot (\rho \nabla q) + (\rho \nabla q) \cdot \nabla p)$,

$$\int ps - \frac{1}{2}g(s,s) = \int \rho \nabla p \cdot \nabla q - \int \rho \frac{1}{2}|\nabla q|^2.$$

Hence

$$\begin{aligned} \frac{1}{2}g(\operatorname{grad} F, \operatorname{grad} F) &= \sup_{q:\mathbf{R}^N \to \mathbf{R}} \left\{ \int \rho \nabla p \cdot \nabla q - \int \rho \frac{1}{2}|\nabla q|^2 \right\} \\ &= \int \rho \frac{1}{2}|\nabla p|^2, \end{aligned}$$

since the maximum is clearly achieved by $q = p$ as evidenced by the fact that the quantity $ax - \frac{1}{2}x^2$ is maximized at $x = a$.

13 Computation of Hess$E$ and Hess$M$

Let $f : \mathcal{M} \to \mathbf{R}$ and $X, Y \in \Gamma$. Then the Hessian is implicitly defined as

$$\operatorname{Hess} f(X,Y) = g(\nabla_X \nabla f, Y).$$

It is easy to check that in Euclidean space this makes sense and corresponds to the usual notion of Hessian. In Otto's notation, we instead denote

$$\operatorname{Hess} f X = \nabla_X \nabla f.$$

Now suppose $G : \mathcal{M} \to \mathbf{R}$ and $\sigma \mapsto \rho(\sigma)$ is a geodesic with

$$\rho(0) = \rho_0 \quad \text{and} \quad \frac{d\rho}{d\sigma}(0) = s,$$

then we note that

$$\begin{aligned} g_{\rho_0}(s, \operatorname{Hess} G_{|\rho_0} s) &= g(s, \nabla_s \nabla G_{|\rho_0}) \\ &= D_s[g(s, \nabla G_{|\rho_0})] - g(\nabla_s s, \nabla G_{|\rho_0}) \\ &= D_s[g(s, \nabla G_{|\rho_0})], \end{aligned}$$

since

$$\nabla_s s = \nabla_s \frac{d\rho}{d\sigma}(0) = \frac{d^2\rho}{d\sigma^2}(0) = 0.$$

Therefore,

$$\begin{aligned} g_{\rho_0}(s, \operatorname{Hess} G_{|\rho_0} s) &= D_s[g(s, \nabla G_{|\rho_0})] \\ &= D_s[\operatorname{diff} G_{|\rho_0}.s] \\ &= \frac{d^2}{d\sigma^2} G(\rho(\sigma))_{|\sigma=0}, \end{aligned}$$

that is, the Hessian of G can be computed by taking second derivatives of G along geodesics. Notice here we have used the fact that $\operatorname{diff} G_{|\rho_0}.s = \frac{d}{d\sigma} G(\rho(\sigma))_{|\sigma=0}$, where $s = \frac{d\rho}{d\sigma}(0)$. This is not difficult to see: We simply compare the expressions

$$\operatorname{diff} G_{|\rho_0}.s = \lim_{t\to 0} \frac{G(\rho_0 + ts) - G(\rho_0)}{t} \quad \text{and} \quad \frac{d}{d\sigma} G(\rho(\sigma))_{|\sigma=0} = \lim_{t\to 0} \frac{G(\rho(0+t)) - G(\rho_0)}{t}$$

and Taylor expand.

Recall that tangent vectors are presented by

$$-\nabla \cdot (\rho_0 \nabla p) = s, \quad g_\rho(s_1, s_2) = \int \rho \nabla p_1 \cdot \nabla p_2$$

and geodesics are characterized as

$$\rho(\sigma) = \nabla\varphi(\sigma) \# \rho_0,$$

where

$$\varphi(\sigma, y) = \frac{1}{2}|y|^2 + \sigma p(y).$$

We can now identify $\operatorname{Hess} M_{|\rho_0}$. We have

$$M(\rho) = \int \frac{1}{2}|y|^2 \rho(y)\ dy = \int \frac{1}{2}|\nabla\varphi_\sigma(x)|^2 \rho_0(x)\ dx = \int \frac{1}{2}\langle x + \sigma\nabla p(x), x + \sigma\nabla p(x)\rangle \rho_0(x)\ dx.$$

Hence

$$\begin{aligned} g_{\rho_0}(s, \operatorname{Hess} M_{|\rho_0} s) &= \frac{d^2}{d\sigma^2}_{|\sigma=0} M(\rho(\sigma)) \\ &= \int \frac{1}{2} \frac{d^2}{d\sigma^2} [\sigma^2 |\nabla p(x)|^2] \rho_0(x)\ dx \\ &= \int |\nabla p|^2 \rho_0 \\ &= g_{\rho_0}(s, s), \end{aligned}$$

and hence (recall ρ_0 is actually arbitrary, as opposed to the ρ_0 which is chosen to be fixed in $\mathcal{M}$

in defining the submersion Φ)

$$\text{Hess}\, M_{|\rho} = \text{id}, \quad \text{for all} \quad \rho \in \mathcal{M}.$$

Much more work is required to compute Hess $E_{|\rho_0}$. By the definition of # we have for all $\zeta \in C_0^\infty(\mathbf{R}^N)$,

$$\int \zeta(\nabla\varphi_\sigma(y))\rho_0(y)\; dy = \int \zeta(x)\rho_\sigma(x)\; dx,$$

and so by the change of variables formula

$$[\det D^2\varphi_\sigma][\rho_\sigma(\nabla\varphi_\sigma(y))] = \rho_0(y),$$

where $D^2\varphi_\sigma$ denotes the $N \times N$ matrix of the second spatial derivatives of φ_σ, i.e. the Hessian. Recalling that $E(\rho) = \int e(\rho)$, we have again by the change of variables formula

$$E(\rho_\sigma) = \int e(\rho_\sigma(x))\; dx = \int e(\rho_\sigma)(\nabla\varphi_\sigma(y))[\det D^2\varphi_\sigma]\; dy,$$

and hence combining with the previous expression, we get

$$E(\rho_\sigma) = \int e\left(\frac{\rho_0}{\det D^2\varphi_\sigma}\right)[\det D^2\varphi_\sigma].$$

Now let us observe three things about $D^2\varphi_\sigma$:

- $D^2\varphi_\sigma$ is symmetric by equality of mixed partials.
- $D^2\varphi_\sigma$ is positive definite for sufficiently small σ, since $D^2\varphi(0) = \frac{1}{2}D^2|y|^2 = \text{id}$.
- $\frac{\partial^2}{\partial\sigma^2}D^2\varphi_\sigma = \frac{1}{2}D^2\frac{\partial^2}{\partial\sigma^2}(|y|^2 + \sigma p(y)) = 0$.

We first aim to show that $\frac{d^2}{d\sigma^2}E(\rho_\sigma) \geq 0$ (and hence E is convex on $\mathcal{M}$). For simplicity of notation, let $\sigma \mapsto A_\sigma$ be a curve in the space of *symmetric* and *positive definite* $N \times N$–matrices which satisfy

$$A(0) = \text{id} \quad \text{and} \quad \frac{d^2A}{d\sigma^2} = 0$$

(we envision $A_\sigma = D^2\varphi_\sigma$) and let $z > 0$ (we envision $z = \rho_0(y)$). Recall the osmotic pressure π is related to the energy density e by

$$\pi(\tilde{z}) = \tilde{z}e'(\tilde{z}) - e(\tilde{z}),$$

so explicit computations show that

$$\frac{d}{d\sigma}\left[e\left(\frac{z}{\det A}\right)\det A\right] = -\pi\left(\frac{z}{\det A}\right)\frac{d}{d\sigma}\det A,$$

$$\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\det A\right] = \pi'\left(\frac{z}{\det A}\right)\frac{z}{(\det A)^2}\left(\frac{d}{d\sigma}\det A\right)^2 - \pi\left(\frac{z}{\det A}\right)\frac{d^2}{d\sigma^2}\det A.$$

By Jacobi's formula, we have that

$$d\det(A) = \mathrm{tr}(\mathrm{adj}(A)dA),$$

where $\mathrm{adj}(A)$ denotes the adjugate of A (conjugate of the matrix formed by cofactors of A), so that

$$\frac{\partial \det(A)}{\partial A_{ij}} = \mathrm{adj}(A)_{ji} = \det(A)(A^{-1})_{ji}$$

(Jacobi's formula can be derived from Laplace's formula for the determinant of a matrix: $\det(A) = \sum_j A_{ij}\mathrm{adj}^T(A)_{ij}$ and the fact that $\sum_i \sum_j A_{ij}B_{ij} = \mathrm{tr}(A^T B)$), so it follows that

$$\frac{d}{d\sigma}\det A = \mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)\det A$$

and

$$\begin{aligned}\frac{d^2}{d\sigma^2}\det A &= \left(\mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)\right)^2\det A + \frac{d}{d\sigma}\left[\mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)\right]\cdot\det A \\ &= \left(\mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)\right)^2\det A + \left[\mathrm{tr}\left(A^{-1}\frac{d^2A}{d\sigma^2}\right) - \mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)^2\right]\cdot\det A.\end{aligned}$$

To establish the last inequality we need to show that

$$\frac{d}{d\sigma}\left[\mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)\right] = \mathrm{tr}\left(A^{-1}\frac{d^2A}{d\sigma^2}\right) - \mathrm{tr}\left(A^{-1}\frac{dA}{d\sigma}\right)^2.$$

In general if P, Q are matrices,

$$\frac{d}{d\sigma}\mathrm{tr}(PQ) = \sum_k \frac{d}{d\sigma}(PQ)_{kk} = \sum_k\sum_\ell \frac{d}{d\sigma}(P_{k\ell}\cdot Q_{\ell k}) = \mathrm{tr}\left(\frac{d}{d\sigma}(PQ)\right),$$

where $\frac{d}{d\sigma}(PQ)$ satisfies the usual product rule. Since as for scalars we have

$$\frac{d(AA^{-1})}{d\sigma} = 0,$$

it follows that

$$\frac{dA^{-1}}{d\sigma} = -A^{-1}\frac{dA}{d\sigma}A^{-1},$$

so

$$\frac{d}{d\sigma}\left(A^{-1}\frac{dA}{d\sigma}\right) = \frac{dA^{-1}}{d\sigma}\frac{dA}{d\sigma} + A^{-1}\frac{d^2A}{d\sigma^2} = A^{-1}\frac{d^2A}{d\sigma^2} - \left(A^{-1}\frac{dA}{d\sigma}\right)^2.$$

The result now follows by taking the trace.

Plugging these expression in, we get

$$\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\det A\right] = \det A\cdot\{(w\pi'(w)-\pi(w))\cdot(\mathrm{tr}(A^{-1}B))^2 + \pi(w)\cdot\left(\mathrm{tr}(A^{-1}B)^2 + \mathrm{tr}\left(A^{-1}\frac{dB}{d\sigma}\right)\right)\},$$

where

$$B = \frac{dA}{d\sigma} \quad \text{and} \quad w = \frac{z}{\det A}.$$

We will now *invoke the condition* that $\frac{d^2A}{d\sigma^2} = 0$ so that the term containing $\frac{dB}{d\sigma}$ is identically zero.

To handle the remaining terms let us set

$$C := A^{-1/2}BA^{-1/2}$$

so that

$$(A^{-1}B)^2 = A^{-1/2}C^2A^{1/2} \quad \implies \mathrm{tr}(A^{-1}B)^2 = \mathrm{tr}(C^2).$$

Since A is symmetric, B is also symmetric, and it is not difficult to see (via e.g., explicit *diagonalization*) that $A^{-1/2}$ is symmetric as well and so C is symmetric. It follows that

$$\mathrm{tr}(C^2) = \sum_k\sum_\ell C_{k\ell}^2 \geq \sum_k C_{kk}^2.$$

On the other hand,

$$(\mathrm{tr}C)^2 = \left(\sum_k C_{kk}\right)^2,$$

so it follows from *convexity* of the *function* $x \mapsto x^2$ that

$$\mathrm{tr}(A^{-1}B)^2 = \mathrm{tr}(C^2) \geq \frac{1}{N}(\mathrm{tr}C)^2 = (\mathrm{tr}(A^{-1}B))^2,$$

and hence (recalling $\pi(z) = z^m$)

$$\begin{aligned}\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\det A\right] &\geq (w\pi'(w) - \left(1-\frac{1}{N}\right)\pi(w))(\mathrm{tr}(A^{-1}B))^2\det A \\ &= (m-(1-\frac{1}{N}))w^m(\mathrm{tr}(A^{-1}B))^2\det A \\ &\geq 0,\end{aligned}$$

where in the last line we have *used the condition* that $m \geq 1-\frac{1}{N}$ and also the fact that $\det(A(0)) = 1$ since $A(0) = \mathrm{id}$, so $\det A \geq 0$ for sufficiently small σ. Recalling $E(\rho_\sigma) = \int e\left(\frac{\rho_0}{\det D^2\varphi_\sigma}\right)\det D^2\varphi_\sigma$, we have shown

$$g_{\rho_0}(s, \mathrm{Hess}\, E_{|\rho_0}s) = \frac{d^2}{d\sigma^2}_{|\sigma=0}E(\rho_\sigma) \geq 0.$$

We have already shown that $g_{\rho_0}(s, \text{Hess}\, M_{|\rho_0}, s) = g_{\rho_0}(s,s)$, so

$$\begin{aligned} g_\rho(s, \text{Hess}\, F_{|\rho} s) &= g_\rho(s, \text{Hess}\, E_{|\rho_0} s) + g_\rho(s, \text{Hess}\, M_{|\rho_0} s) \\ &\geq |s|^2, \end{aligned}$$

that is, F is uniformly strictly convex on $(\mathcal{M}, g)$. We note that a more explicit expression for $\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\det A\right]$ at $\sigma = 0$ is possible using the fact that $A(0) = \text{id}$.

14 Derivation of Main Results by Formal Riemannian Calculus

Let us now formally derive the main results from the following ingredients (we again denote $\hat\rho = \rho$):

- $\frac{d\rho}{dt} = -\text{grad}\, F_{|\rho}$
- $-\text{grad}\, F_{|\rho_*} = 0$
- $\langle s, \text{Hess}\, F_{|\rho} s\rangle \geq \alpha |s|^2$

First we derived that

$$\frac{d}{d\tau}(e^{2\alpha\tau}|\text{grad}\, F_{|\rho}|^2) \leq 0.$$

We have (with $\frac{D}{d\tau}$ denoting covariant derivative)

$$\begin{aligned} \frac{d}{d\tau}|\text{grad}\, F_{|\rho}|^2 &= 2g\left(\text{grad}\, F_{|\rho}, \frac{D}{d\tau}\text{grad}\, F_{|\rho}\right) \\ &= 2g\left(\text{grad}\, F_{|\rho}, \text{Hess}\, F_{|\rho}\frac{d\rho}{d\tau}\right) \\ &= -2g(\text{grad}\, F_{|\rho}, \text{Hess}\, F_{|\rho}\text{grad}\, F_\rho) \\ &\leq -2\alpha|\text{grad}\, F_\rho|^2, \end{aligned}$$

hence if we set $G(\tau) = |\text{grad}\, F_{|\rho}|^2$, then $\frac{dG}{d\tau} \leq -2\alpha G$ so that $\frac{d}{d\tau}(e^{2\alpha\tau}G) = e^{2\alpha\tau}(2\alpha G + \frac{dG}{d\tau}) \leq 0$.

Next we bound F by $\text{grad}\, F$ and $\text{grad}\, F$ by $\text{Hess}\, F$. First we let $\sigma \mapsto \rho(\sigma)$ be a curve of least energy, so that

$$d(\rho_0, \rho_1)^2 = \int_0^1 \left|\frac{d\rho}{d\sigma}\right|^2\, d\sigma.$$

Recall this means that $\sigma \mapsto \rho(\sigma)$ is a geodesic so that

$$\frac{D}{d\sigma}\frac{d\rho}{d\sigma} = 0,$$

and so the speed is also constant:

$$\frac{d}{d\sigma}\left|\frac{d\rho}{d\sigma}\right|^2 = 2g\left(\frac{d\rho}{d\sigma}, \frac{D}{d\sigma}\frac{d\rho}{d\sigma}\right) = 0.$$

Now by definition of grad, we have

$$\frac{d}{d\sigma}F(\rho) = \operatorname{diff} F(\rho).\frac{d\rho}{d\sigma} = g\left(\frac{d\rho}{d\sigma}, \operatorname{grad} F_{|\rho}\right),$$

so that by the "metric" property of the covariant derivative

$$\begin{aligned}
\frac{d^2}{d\sigma^2}F(\rho) &= g\left(\frac{D}{d\sigma}\frac{d\rho}{d\sigma}, \operatorname{grad} F_{|\rho}\right) + g\left(\frac{d\rho}{d\sigma}, \frac{D}{d\sigma}\operatorname{grad} F_{|\rho}\right)\\
&= g\left(\frac{d\rho}{d\sigma}, \operatorname{Hess} F_{|\rho}\frac{d\rho}{d\sigma}\right)\\
&\geq \alpha \left|\frac{d\rho}{d\sigma}\right|^2\\
&= \alpha\ d(\rho_0,\rho_1)^2.
\end{aligned}$$

Viewing $F(\rho)$ as a real–valued function of σ and Taylor expanding, we obtain that

$$\begin{aligned}
F(\rho_1) - F(\rho_0) &= \frac{d}{d\sigma}F(\rho)_{|\sigma=0} + \frac{1}{2}\frac{d^2}{d\sigma^2}F(\rho)_{|\sigma=0} + \dots\\
&\geq g\left(\frac{d\rho}{d\sigma}_{|\sigma=0}, \operatorname{grad} F_{|\rho_0}\right) + \alpha\frac{1}{2}d(\rho_0,\rho_1)^2.
\end{aligned}$$

Similarly,

$$F(\rho_0) - F(\rho_1) \geq -g\left(\frac{d\rho}{d\sigma}_{|\sigma=1}, \operatorname{grad} F_{|\rho_1}\right) + \alpha\frac{1}{2}d(\rho_0,\rho_1)^2.$$

Adding the previous two displays, we obtain

$$g\left(\frac{d\rho}{d\sigma}_{|\sigma=1}, \operatorname{grad} F_{|\rho_1}\right) - g\left(\frac{d\rho}{d\sigma}_{|\sigma=0}, \operatorname{grad} F_{|\rho_0}\right) \geq \alpha\ d(\rho_0,\rho_1)^2.$$

Notice also that an application of Cauchy–Schwarz implies that

$$F(\rho_1) - F(\rho_0) \geq -\left|\frac{d\rho}{d\sigma}_{|\sigma=0}\right| |\operatorname{grad} F_{|\rho_0}| = -d(\rho_0,\rho_1)|\operatorname{grad} F_{|\rho_0}|.$$

By considering $F(\rho_0) - F(\rho_1)$ and combining, we get

$$|F(\rho_1) - F(\rho_0)| \leq d(\rho_0,\rho_1)\max\{|\operatorname{grad} F_{|\rho_0}|, |\operatorname{grad} F_{|\rho_1}|.$$

Now we can derive

$$\frac{d}{d\tau}(e^{2\alpha\tau}d(\rho,\rho_*)^2) \leq 0.$$

Recall that $F(\rho) - F(\rho_*) \geq 0$ so that ρ_* is a minimizer of F and hence

$$0 = -\operatorname{grad} F_{\rho_*},$$

so that $\rho(\tau) = \rho_*$ is a stationary solution of the evolution equation $\frac{d\rho}{d\tau} = -\mathrm{grad}\, F_\rho$, so it is enough to show that if $\rho_0(\tau)$ and $\rho_1(\tau)$ are two solutions of the evolution equation then we have the contraction property

$$\frac{d^+}{d\tau} d(\rho_1, \rho_0)^2 \leq 2\alpha\ d(\rho_1, \rho_0)^2.$$

where $\frac{d^+}{d\tau}$ denotes derivative from the right, i.e.

$$\frac{d^+}{d\tau}_{|\tau_0} f = \limsup_{\tau \downarrow \tau_0} \frac{f(\tau) - f(\tau_0)}{\tau - \tau_0}.$$

To this end let us fix τ_0 and for any τ we let $\sigma \mapsto \rho(\tau, \sigma) \in \mathcal{M}$ be a curve between $\rho(\tau, 0) = \rho_0(\tau)$ and $\rho(\tau, 1) = \rho_1(\tau)$. We take this to be the curve of least energy at $\tau = \tau_0$ and continuous in τ so that

$$d(\rho_1(\tau), \rho_0(\tau)) = \int_0^1 \left|\frac{\partial \rho}{\partial \sigma}\right|^2\ d\sigma$$

for any τ, with equality at $\tau = \tau_0$. This implies in particular that

$$\frac{d^+}{d\tau}_{|\tau_0} d(\rho_1, \rho_0)^2 \leq \frac{d}{d\tau}_{|\tau_0} \int_0^1 \left|\frac{\partial \rho}{\partial \sigma}\right|^2\ d\sigma = 2\int_0^1 g\left(\frac{\partial \rho}{\partial \sigma}, \frac{D}{d\tau}_{|\tau_0} \frac{\partial \rho}{\partial \sigma}\right)\ d\sigma.$$

Interchanging the σ and τ partial derivatives and using the metric property of the covariant derivative, the above expression is seen to be equal to

$$2\int_0^1 \left\{\frac{d}{d\sigma} g\left(\frac{\partial \rho}{\partial \sigma}, \frac{\partial \rho}{\partial \tau}_{|\tau_0}\right) - g\left(\frac{D}{d\sigma}\frac{\partial \rho}{\partial \sigma}, \frac{\partial \rho}{\partial \tau}_{|\tau_0}\right)\right\}\ d\sigma.$$

Since $\rho(\sigma, \tau_0)$ is a curve of least energy, $\frac{D}{d\sigma}\frac{\partial \rho}{\partial \sigma} = 0$ and so the second term is zero and now we can continue (using the fundamental theorem of calculus) the expression as

$$\begin{aligned} 2\int_0^1 \frac{d}{d\sigma} g\left(\frac{\partial \rho}{\partial \sigma}, \frac{\partial \rho}{\partial \tau}_{|\tau_0}\right)\ d\sigma &= 2\left[g\left(\frac{d\rho}{d\sigma}_{|\sigma=1}, \frac{d\rho_1}{d\tau}\right) - g\left(\frac{d\rho}{d\sigma}_{|\sigma=0}, \frac{d\rho_0}{d\tau}\right)\right] \\ &= -2\left[g\left(\frac{d\rho}{d\sigma}_{|\sigma=1}, \mathrm{grad}\, F_{|\rho_1}\right) - g\left(\frac{d\rho}{d\sigma}_{|\sigma=0}, \mathrm{grad}\, F_{|\rho_0}\right)\right] \\ &\leq -2\alpha\ d(\rho_0, \rho_1)^2. \end{aligned}$$

(Note that in the last three lines everything is evaluated at τ_0).

Finally, we get that

$$\frac{d}{d\tau}(e^{2\alpha\tau}(F(\rho) - F(\rho_*))) \leq 0.$$

Indeed, we already have

$$\lim_{\tau \uparrow \infty} d(\rho, \rho_*) = 0 \quad \text{and} \quad \lim_{\tau \uparrow \infty} |\mathrm{grad}\, F_{|\rho}| = 0,$$

(here of course $|\mathrm{grad}\, F_{|\rho}| = \sqrt{g\left(\mathrm{grad}\, F_{|\rho}, \mathrm{grad}\, F_{|\rho}\right)}$) so using $\mathrm{grad}\, F_{|\rho_*} = 0$, we have

$$|F(\rho) - F(\rho_*)| \leq |\mathrm{grad}\, F_{|\rho}| d(\rho, \rho_*),$$

for all $\rho \in \mathcal{M}$, and hence

$$\lim_{\tau \uparrow \infty} (F(\rho) - F(\rho_*)) = 0.$$

Now let's compute:

$$\begin{aligned}
\frac{d}{d\tau}(F(\rho) - F(\rho_*)) &= g\left(\mathrm{grad}\, F_{|\rho}, \frac{d\rho}{d\tau}\right) - g\left(\mathrm{grad}\, F_{\rho_*}, \frac{d\rho}{d\tau}\right) \\
&= -|\mathrm{grad}\, F_{|\rho}|^2 \\
&= \int_\tau^\infty \frac{d}{d\tau} |\mathrm{grad}\, F_{|\rho}|^2 \, d\tau \\
&\leq -2\alpha \int_\tau^\infty |\mathrm{grad}\, F_{|\rho}|^2 \, d\tau \\
&= 2\alpha \int_\tau^\infty \frac{d}{d\tau}(F(\rho) - F(\rho_*)) \, d\tau \\
&= -2\alpha(F(\rho) - F(\rho_*)).
\end{aligned}$$

Remark. The explicit computations of the gradient and Hessian will be later used to mimic the Riemannian calculus in the smooth setting.

This is a compilation of notes on the paper [3] by Felix Otto, for a talk in the nonlinear PDE seminar in F'08 run by I. Kim at UCLA (we thank Inwon for some useful discussions). We have permuted the order in which some topics appear; in particular, we opted for a complete description of the geometry of $\mathcal{M}$ before stating and deriving the main results. For pedagogical purposes and convenience, we have been repetitive in some places. We have attempted to offer more detailed or slightly different explanations of the results in [3] where we can – especially elaborating a little more on the geometric concepts involved, but on the other hand, some segments are taken almost verbatim from [3].

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References

[1] Y. Du. *Order Structure and Topological Methods in Nonlinear Partial Differential Equations: Maximum Principles and Applications, Volume 1.* World Scientific (2006).

[2] J. M. Lee. *Introduction to Smooth Manifolds.* Springer–Verlag New York, Inc. (2003).

[3] F. Otto. *The Geometry of Dissipative Evolution Equations: The Porous Medium Equation.* Comm. in Partial Differential Equations, 26 (1&2), 101–174 (2001.

[4] P. Petersen. *Riemannian Geometry.* Springer Science+Business Media, LLC, New York (2006).

[5] T. Tao. Lecture notes on manifolds/Riemannian manifolds found at: http://terrytao.wordpress.com/2008/03/26/285g-lecture-0-riemannian-manifolds-and-curvature/

[6] J. L. Vazquez. *Two Nonlinear Diffusion Equations with Finite Speed of Propagation.* Lecture Notes in Physics, Springer Berlin / Heidelberg, Vol. 359 197–206 (1990).

[7] J. L. Vazquez. *The Porous Medium Equation.* Oxford Mathematical Monographs, Oxford University Press, US (2007).