Some Notes on Field Theory Preliminaries

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Based on the following references:

- *Gaussian Hilbert Spaces* by Svante Janson.
  Cambridge University Press (1997).
- *Gaussian free field and conformal field theory* by N. Kang and N. Makarov.
  Available at http://arxiv.org/abs/1101.1024.
- *Gaussian free fields for mathematicians* by S. Sheffield.
  Available at http://arxiv.org/abs/math/0312099.

Wick's Multiplication

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Most of the results we explain below can be found in e.g., *Gaussian Hilbert Spaces* by Svante Janson. We will write $X \sim \mathcal{N}(\mu, \sigma)$ to denote that X is a Gaussian with mean μ and variance σ. In general, if X and Y are random variables, $X \sim Y$ means they have the same law.

1 Gaussian

- A real valued random variable $X : (\Omega, \mathbf{P}) \to \mathbb{R}$ is Gaussian with mean μ and variance σ if $X\#\mathbf{P} = f(x)\ dx$, where

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

  To check e.g., that the mean is μ, we have

$$\begin{aligned} \mathbf{E}X &= \int_\Omega X(\omega)\ d\mathbf{P}(\omega) \\ &= \int_{\mathbb{R}} x f(x)\ dx \\ &= ... \end{aligned}$$

- The characteristic function of a random variable X is given as $\mathbf{E}(e^{iXt})$. If X Gaussian with mean μ and variance σ^2, then

$$\begin{aligned} \mathbf{E}(e^{itX}) &= \frac{1}{\sqrt{2\pi}\sigma} \int e^{itx} e^{-(x-\mu)^2/2\sigma^2}\ dx \\ &= e^{i\mu t - \frac{1}{2}\sigma^2 t^2}, \end{aligned}$$

  where the computation follows by completing the square in the exponential and using the fact that

$$\int e^{-x^2/2}\ dx = \sqrt{\left(\int e^{-x^2/2}\ dx\right) \cdot \left(\int e^{-y^2/2}\ dy\right)} = \sqrt{\left(\int_0^{2\pi} \int_{\mathbb{R}} e^{-r^2/2} r\ dr\ d\theta\right)} = \sqrt{2\pi}.$$

  Notice that this easily generalize to higher dimensions:

$$\int_{\mathbb{R}^d} e^{-|x|^2/2}\ dx = \left(\int e^{-x_1^2/2}\ dx^1\right) \cdots \left(\int e^{-x_d^2/2}\ dx^d\right) = (2\pi)^{d/2}.$$

As an aside, we can use this to derive an expression for the surface area of the $d-1$–sphere. To start we will change to spherical coordinates by writing

$$x = r\omega, \quad r = |x|, \quad \frac{x}{|x|} = \omega = (\omega_1, \dots, \omega_d) \in S^{d-1},$$

so that $\omega_d = \pm\sqrt{1-(\omega_1^2+\cdots+\omega_{d-1}^2)}$. Since the formulas will clearly generalize, for simplicity we take $d=3$. We have

$$\begin{aligned} dx^1 &= r\ d\omega^1 + \omega_1\ dr \\ dx^2 &= r\ d\omega^2 + \omega_2\ dr \\ dx^3 &= r\ d\omega^3 \pm \sqrt{1-(\omega_1^2+\omega_2^2)}\ dr \\ &= \pm r \cdot \left(\frac{\omega_1}{\sqrt{1-(\omega_1^2+\omega_2^2)}}\ d\omega_1 + \frac{\omega_2}{\sqrt{1-(\omega_1^2+\omega_2^2)}}\ d\omega_2 \right) \pm \sqrt{1-(\omega_1^2+\omega_2^2)}\ dr \end{aligned}$$

Thus (we recall that $dx \wedge dx = 0$, $dx \wedge dy = -dy \wedge dx$)

$$\begin{aligned} dx^1 \wedge dx^2 \wedge dx^3 &= r^2 \omega_3^+\ d\omega^1 \wedge d\omega^2 \wedge dr + r^2 \frac{\omega_2^2}{\omega_3^+}\ d\omega^1 \wedge dr \wedge d\omega^2 + r^2 \frac{\omega_1^2}{\omega_3^+}\ dr \wedge d\omega^2 \wedge d\omega^1 \\ &= \pm \left[\frac{r\ d\omega^1 \wedge d\omega^2}{\sqrt{1-(\omega_1^2+\omega_2^2)}} \wedge r\ dr \right] \\ &= dS_2 \wedge r\ dr. \end{aligned}$$

Of course there is a more symmetrized expression (note also the singularity at the poles) since writing ω_3 in terms of ω_1 and ω_2 is arbitrary. In any case, it is clear that dS_2 represents the surface area element and there therefore, denoting the area of the unit d–sphere by $A(d)$ and denoting by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\ dt$$

the Gamma function, we have

$$\begin{aligned} (2\pi)^{2/d} = \int_{\mathbb{R}^d} e^{-|x|^2/2}\ dx &= \int_0^\infty \int_{S^d} dS^d\ e^{-r^2/2} r^{d-1}\ dr \\ &= 2^{d/2-1} A(d) \int_0^\infty e^{-u} u^{d/2-1}\ du = 2^{d/2-1} A(d) \Gamma(d/2), \end{aligned}$$

from which we conclude that

$$A(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

- Consider $\mathbb{R}^d$ equipped with the usual inner product, and let

$$\mu = \frac{1}{(2\pi)^{d/2}} \cdot e^{-\langle x,x\rangle/2}\ dx$$

be the normalized d–dimensional (centered) Gaussian measure and let X be a random variable (vector) on $\mathbb{R}^d$, so that the characteristic function is

$$\mathbb{E}(e^{i\langle X,t\rangle}) = \int_{\mathbb{R}^d} e^{i\langle x,t\rangle}\ \frac{1}{(2\pi)^{1/2}} \cdot e^{-\langle x,x\rangle/2}\ dx = e^{-\langle t,t\rangle/2}.$$

The following are equivalent:

1. X has law μ.
2. Let $\{v_j\}_{j=1}^d$ be an orthonormal basis and let $\alpha^j \sim \mathcal{N}(0,1)$ be i.i.d., then

$$X \sim \sum_{j=1}^{d} \alpha^j v_j,$$

3. For each fixed $w \in \mathbb{R}^d$,

$$\langle X, w\rangle \sim \mathcal{N}(0, \langle w, w\rangle).$$

That 1) is equivalent to 2) is clear by a change of basis: By independence, the density function for $\sum_{j=1}^d \alpha^j v_j$ explicitly takes the same form as μ. To see that 3) equivalent to 1) it is sufficient to compute the relevant characteristic functions (if X and Y have the same characteristic functions, then their distributions have the same Fourier transform).

While the above is relatively elementary, some remarks are in order:

- Physically, a random field can be thought of as a random "function" which describes the fluctuation at each point. In this way a field is then a mapping $x \mapsto X_x$, where $X : \Omega \to \mathbb{R}$ is some random variable, that is, a field is a sequence of random variables indexed by $\mathbb{R}^d$.
- From the above, we see that given a measure μ on $\mathbb{R}^d$, there is a natural way to obtain such a mapping: We consider a random vector $X \sim \mu$ and then given a point $w \in \mathbb{R}^d$, we consider the projection of X onto w, the random variable $\langle X, w\rangle$.
- From this we also see that the random field should be viewed as being on the space dual to the "physical" space. Here X acts on $\mathbb{R}^d$ as a random linear functional: $\omega \mapsto \langle X(\omega), w\rangle \in \mathbb{R} \times \mathbb{R}^d$ and letting $\{v_1, \ldots, v_d\}$ be an orthonormal basis for $\mathbb{R}^d$, it is clear that

$$\langle X, e_k\rangle \# P = \text{Distrib}(\pi_k \cdot \mathbb{R}^d).$$

  Here π_k denotes the canonical projection operator.
- Reversing arrows, we have that $w \in \mathbb{R}^d$ acts on $L^2(\Omega, P)$ (some space of random variables) as a random linear functional: Given $w \in \mathbb{R}^d$, consider the mapping $\omega \mapsto \mathcal{F}(\omega) := \{\langle w, X(\omega)\rangle : X \in L^2(\Omega)\} \in \mathbb{R} \times L^2(\Omega)$, then

$$\langle w, \cdot\rangle : \Omega \to \mathbb{R} \times L^2(\Omega) : \omega \mapsto \mathcal{F}(\omega)$$

  such that

$$\langle X, w\rangle \# P = \text{Distrib}(\pi_X \cdot L^2(\Omega)).$$

  Note that here we are not specific about the structure of $L^2(\Omega)$.
- Alternatively (as is usually done) we may take a Gaussian–centric perspective: There is only one random variable which is Gaussian and the local differences in physical space is described by some function in a suitable function space. In this way, we consider e.g., a Hilbert space of functions and thus, imposing a commensurability of structures condition, we arrive at the definition of a *Gaussian field*:

  A Gaussian field indexed by a Hilbert space $\mathcal{H}$ is an isometry $\Phi : \mathcal{H} \to L^2(\Omega, P)$ such that the image are centered Gaussian.

As a particular example, suppose $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis for $\mathcal{H}$, and $e_\alpha \mapsto \xi^\alpha$ with $\xi^\alpha \sim \mathcal{N}(0,1)$ independent, then clearly,

$$\langle \xi^\alpha, \xi^\alpha \rangle_{L^2(\Omega,P)} = \mathbf{E}|\xi^\alpha|^2 = \sigma^2 + \mu^2 = 1 = \langle e_\alpha, e_\alpha \rangle_{\mathcal{H}}.$$

This is a generalization of 2) from above: *Formally*, we can represent this isometry by considering $X \sim \sum_{\alpha\in A} \xi^\alpha e_\alpha$, then given $f \in \mathcal{H}$, $f \mapsto \langle X, f\rangle = \sum_{\alpha\in A} \xi^\alpha f^\alpha$.

It is noted that the definition implicates that the relevant portion of the space $L^2(\Omega, P)$ is itself a Hilbert space (a *Gaussian Hilbert space*). By direct computation we can see that if X, Y are independent centered Gaussian, then so is $X + Y$ (we recall the distribution function identity $f_{X+Y}(a) = (f_X * f_Y)(a)$ where $*$ denotes convolution). Otherwise, we say $X_1, \dots, X_n$ are *jointly* Gaussian if any linear combination of them is Gaussian.

- Next we note *Wick's formula*: If $X_1, \dots, X_n$ are jointly Gaussian with mean zero, then

$$\mathbf{E}[X_1 \dots X_n] = \sum \prod \mathbf{E}[X_i X_j],$$

where the sum is over pair partitions, so that in particular the above is zero if n is odd. The formula follows either by explicit power series expansion of the joint characteristic function $\mathbf{E}(e^{\sum_{k=1}^n t_k X_k})$ and noting that odd moments of a centered Gaussian vanish by symmetry or by noting that both sides are multilinear forms on a (Gaussian) vector space (if $X_1, \dots, X_n$ are jointly Gaussian, then they can be embedded in some Gaussian vector space) and using polarization. We prefer the latter because of its combinatorial content.

We will denote by $\langle \dots \rangle^{(n)}$ a generic n–linear form and will suppress repeated arguments where it is clear. We first establish the formula that

$$\langle x_1, \dots, x_n \rangle = \frac{1}{2^n n!} \sum_{\sigma=(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n} \operatorname{sgn}(\sigma) \langle (\sigma \cdot \vec{x})^{(n)} \rangle^{(n)}.$$

(Here the argument $\sigma \cdot \vec{x} = \sigma \cdot (x_1, \dots, x_n)$ is repeated n times.) When $n = 2$, this is the usual polarization formula:

$$\langle x, y \rangle = \frac{1}{2}(\langle x + \frac{y}{2}, x + \frac{y}{2} \rangle - \langle x - \frac{y}{2}, x - \frac{y}{2} \rangle).$$

We can now proceed by induction. First we observe that if $\{e_\alpha\}$ is an orthonormal basis for the vector space, then we may define the $n-1$–linear forms by

$$\langle x_1, \dots, x_{n-1} \rangle_\alpha^{(n-1)} := \langle x_1, \dots, x_{n-1}, e_\alpha \rangle^{(n)},$$

so that if $x_n = x_n^\alpha e^\alpha$, then

$$\langle x_1, \dots, x_{n-1}, x_n \rangle^{(n)} = x_n^\alpha \cdot \langle x_1, \dots, x_{n-1} \rangle_\alpha^{(n-1)}.$$

(Here we have used Einstein summation convention to suppress the sum over α.) Next it is observed that by multi–linearity we may symmetrize as follows:

$$\langle x^{(n-1)}, x_n \rangle^{(n)} = \frac{1}{2} \cdot \frac{1}{n} \cdot \langle (x + x_n)^{(n)} \rangle^{(n)} - \langle (x - x_n)^{(n)} \rangle,$$

where the factor of $1/n$ is due to the fact that the term x_n can occur in n places and because of symmetry the result is the same. We may now conclude:

$$\begin{aligned}
\langle x_1, \ldots, x_n \rangle^{(n)} &= x_n^\alpha \cdot \langle x_1, \ldots, x_{n-1} \rangle_\alpha^{(n-1)} \\
&= x_n^\alpha \cdot \frac{1}{2^{n-1}} \frac{1}{(n-1)!} \sum_{\sigma'=(\varepsilon_1,\ldots,\varepsilon_{n-1})\in\{\pm 1\}^{n-1}} \operatorname{sgn}(\sigma') \langle (\sigma' \cdot (x_1, \ldots, x_{n-1})) \rangle_\alpha^{(n-1)} \\
&= \frac{1}{2^{n-1}} \frac{1}{(n-1)!} \sum_{\sigma'} \operatorname{sgn}(\sigma') \cdot \langle (\sigma' \cdot (x_1, \ldots, x_{n-1}))^{(n-1)}, x_n \rangle^{(n)} \\
&= \frac{1}{2^{n-1}} \frac{1}{(n-1)!} \sum_{\sigma'} \operatorname{sgn}(\sigma') \cdot \frac{1}{2} \cdot \frac{1}{n} \times \\
&\qquad \langle [\sigma' \cdot (x_1, \ldots, x_{n-1}) + x_n]^{(n)} \rangle^{(n)} - \langle [\sigma' \cdot (x_1, \ldots, x_{n-1}) - x_n]^{(n)} \rangle^{(n)} \\
&= \frac{1}{2^n} \frac{1}{n!} \sum_{\sigma=(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm 1\}^n} \langle (\sigma \cdot (x_1, \ldots, x_n))^{(n)} \rangle^{(n)}.
\end{aligned}$$

The result implies in particular that two multi–linear forms are the same if they are the same evaluated on a diagonal. Thus it is sufficient to check that if X is a centered Gaussian with variance σ^2, then for n even,

$$\mathbf{E}X^n = (n-1)!!\ (\mathbf{E}X^2)^{n/2} = (n-1)!!\ \sigma^n.$$

Here $(n-1)!! = (n-1) \cdot (n-3) \cdot \cdots \cdot 1$ is the number of pair partitions of n. A computation as before gives that the moment generating function is $\mathbf{E}(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\frac{1}{2}\sigma^2 t^2}$ and the result follows, since $\mathbf{E}X^n$ is $n!$ times the coefficient of t^n which is $\left(\frac{n}{2}\right)! \left(\frac{1}{2}\right)^{n/2}$ and we have the identity

$$\frac{n!}{(n/2)!} \left(\frac{1}{2}\right)^{n/2} = (n-1)!!$$

(This easily follows by induction or by counting in two ways: The left hand side follows by permuting all n elements and putting bars between every two and dividing by overcounting (or "symmetries") whereas the right hand side follows by choosing the pairs one by one.)

Some remarks:

- What is obvious: A centered Gaussian random variable is completely determined by its second moment, or σ^2. Therefore, determination of the field Φ is equivalent to the determination of $\mathbf{E}(|\Phi(f)|^2)$ for all $f \in \mathcal{H}$. In the simplest case we may expect this to be represented by some functional $L_\Phi^{(2)} : \mathcal{H} \to \mathbb{R}$. In the familiar setting where $\mathcal{H}$ is some function space equipped with a measure and a notion of integration, we may even expect $L_\Phi^{(2)}$ to be represented "distributionally":

$$L_\Phi^{(2)}(f) = \int L_\Phi^{(2)}(z) f(z)\ d\mu(z).$$

  This foreshadows the definition of correlation functions to come.

- Since only even order moments and covariances are relevant in the case of a Gaussian, it is useful to use *Feynman diagrams* in visualizing quantities like the formula for $\mathbf{E}[X_1, \ldots, X_n]$: A Feynman diagram on n vertices is a degree ≤ 1 graph, where the vertices represent random variables $\{X_1, \dots, X_n\}$ and the existence of an edge between vertex k and vertex ℓ represents the quantity $\mathbf{E}[X_k X_\ell]$. Denoting a Feynman diagram by

$$\gamma := e(\gamma) \cup v(\gamma),$$

where $e(\gamma)$ and $v(\gamma)$ denotes the set of edges and unpaired vertices, respectively, the *value* of the Feynman diagram is then

$$V(\gamma) = \prod_{\langle k,\ell\rangle \in e(\gamma)} \mathbf{E}[X_k X_\ell] \times \prod_{m \in v(\gamma)} X_m.$$

Let us note that the number of Feynman diagrams on n vertices with *rank* k (i.e., $e(\gamma) = k$) is $\binom{n}{2k}(2k-1)!!$

2 Wick's Multiplication and Fock Space

We want some notion of products of fields. Since a Gaussian field is an isometry from a Hilbert space into $L^2(\Omega)$, we shall borrow the algebraic structure from the Hilbert space. Pointwise multiplication of functions (from two function spaces) corresponds to tensor product: $f \otimes g(x) = f(x)g(x)$. The goal is then to construct a commutative product (Wick's product) on Gaussian random variables which respects this pointwise multiplication.

- Tensor product of spaces captures the essence of multi–linearity: Suppose V and W are vector spaces, then $V \otimes W$ can be realized as $V \times W$ modulo bilinear equivalence relations, e.g., we make the identification

  $$(v + v', w + w') \sim (v, w) + (v, w') + (v', w) + (v', w').$$

  This can be encoded as a bilinear map $\tau : V \times W \to V \otimes W : (v, w) \mapsto v \otimes w$.

  - Indeed, $V \otimes W$ can be thought of as the "maximal" bilinear structure on $V \times W$, as captured by the universal property of tensor products: If $\varphi : V \times W \to U$ is bilinear, then $\exists! \psi : V \otimes W \to U$ such that φ can be factor through τ, i.e., $\varphi = \psi \circ \tau$.
  - In e.g., the case of finite dimensional vector spaces, the space of bilinear forms on $V \times W$ is $V^* \otimes W^* \cong V \otimes W$ where the action is given as

    $$v^* \otimes w^*(v, w) = v^*(v)w^*(w).$$

  We note that the dimension of $V \otimes W$ is $n \times m$ if $\dim(V) = n$ and $\dim(W) = m$ (whereas the dimension of $V \times W$ is $n + m$). In particular, an element of $V \otimes W$ can be represented by an $m \times n$ matrix. As an example, a Riemannian metric on $\mathbb{R}^d$ is a bilinear form (more specifically an inner product which varies from point to point) and can be represented by the $d \times d$ coefficients $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$ where $\{\partial_\alpha\}_\alpha$ is a frame for the tangent space. Equivalently, we may expand the 2–form in basis as

  $$g(X^\lambda \partial_\lambda, Y^\mu \partial_\mu) = g_{\alpha\beta}(\theta^\alpha \otimes \theta^\beta)(X^\lambda \partial_\lambda, Y^\mu \partial_\mu) = g_{\alpha\beta} X^\alpha Y^\beta.$$

- If V and W are Hilbert spaces equipped with an inner product, then there is a unique inner product on $V \otimes W$ defined by component–wise application of the V and W inner products:

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{V \otimes W} = \langle v_1, v_2 \rangle_V \langle w_1, w_2 \rangle_W.$$

  It is easy to check that this defines a bilinear form on $V \otimes W$ and that it is well–defined (and hence unique). The *Hilbert space* inner product is then the closure of this inner product.

- $V \otimes V$ is not in general commutative ($v \otimes w \neq w \otimes v$) and in cases where commutativity is present, it is convenient to define the *symmetric tensor product*, which is denoted

$$V \odot W \cong V \otimes W / \langle v \otimes w - w \otimes v \rangle.$$

  (Here $\langle \mathcal{A} \rangle$ denotes the vector space generated by $\mathcal{A}$.) That is, we simply identify $v \otimes w$ with $w \otimes v$. Clearly, this corresponds to symmetric multilinear forms, as e.g., most Riemannian metric are. In the case of Hilbert spaces, the inner product on $V \odot W$ now acquires a combinatorial sum: Since $\langle v_1 \odot v_2, w_1 \odot w_2 \rangle = \langle v_1 \odot v_2, w_2 \odot w_1 \rangle$, we define

$$\langle v_1 \odot v_2, w_1 \odot w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle + \langle v_1, w_2 \rangle \langle v_2, w_1 \rangle.$$

  (Note that we are already using that the original inner product $\langle \cdot, \cdot \rangle$ is symmetric.) More generally, letting S_n denote the permutation group on n letters, we define

$$\langle f_1 \odot f_2 \cdots \odot f_n, g_1 \odot g_2 \odot \cdots \odot g_n \rangle = \sum_{\sigma \in S_n} \prod_{k=1}^{n} \langle f_k, g_{\sigma(k)} \rangle.$$

  Finally, let us note the simple combinatorial fact that

$$\langle \xi^{\odot(n+m)}, \xi^{\odot(n+m)} \rangle = \binom{m+n}{n} \langle \xi^{\odot n}, \xi^{\odot n} \rangle \langle \xi^{\odot m}, \xi^{\odot m} \rangle.$$

- We now try to transfer this product structure from the Hilbert space $\mathcal{H}$ to $L^2(\Omega, P)$ via our isometry $\Phi : \mathcal{H} \to L^2(\Omega, P)$. By definition

$$\Phi(h \odot g) = \Phi(h) \odot \Phi(g) := \xi_h \odot \xi_g,$$

  where ξ_h, ξ_g are $\mathcal{N}(0, \sigma_h), \mathcal{N}(0, \sigma_g)$. We now want to make sense of $\xi_h \odot \xi_g$ in terms of usual products and sums of random variables. Said differently, we clearly have an isometry $\Phi \odot \Phi : \mathcal{H} \odot \mathcal{H} \to L^2 \odot L^2$ but we now would like to find some map $L^2 \odot L^2 \rightsquigarrow L^2$ such that the composition is still an isometry (with respect to the original L^2 inner product).

Let us begin with a computation comparing the corresponding norms of $\xi_h \odot \xi_g$ and $\xi_h \xi_g$:

$$\langle \xi_h \odot \xi_g, \xi_h \odot \xi_g \rangle_{L^2 \odot L^2} = \mathbf{E}\xi_h^2 \mathbf{E}\xi_g^2 + (\mathbf{E}(\xi_h \xi_g))^2 \neq \mathbf{E}(\xi_h^2 \xi_g^2) = \langle \xi_h \xi_g, \xi_h \xi_g \rangle_{L^2} = \mathbf{E}(\xi_h^2 \xi_g^2),$$

unless ξ_h, ξ_g are *independent* random variables, so that

$$\mathrm{Cov}(\xi_h, \xi_g) = \mathbf{E}[\xi_h \xi_g] = 0$$

and

$$\mathrm{Cov}(\xi_h^2, \xi_g^2) = \mathbf{E}[(\xi_h^2 - \sigma_h^2)(\xi_g^2 - \sigma_g^2)] = 0.$$

It is also observed that by the definition of an isometry, if f and g are orthogonal in $\mathcal{H}$, then ξ_f and ξ_g are independent in L^2; that is, orthogonality in $\mathcal{H}$ corresponds to probabilistic independence in L^2. In any case, we may make the identification

$$\xi \odot \eta \approx \xi\eta,$$

provided that they are independent.

We are therefore reduced to the question of representing powers $\xi^{\odot n}$ in a meaningful way. First we have

$$\langle \xi \odot \xi, \xi \odot \xi \rangle = 2\langle \xi, \xi \rangle^2 = 2\sigma_\xi^4,$$

whereas (recall that $\mathbf{E}\xi^n = (n-1)!!\sigma_\xi^n$ for n even)

$$\langle \xi^2, \xi^2 \rangle = \mathbf{E}\xi^4 = 3\sigma_\xi^4.$$

It is trivial to renormalize appropriately, or e.g., try to represent $\xi^{\odot n}$ as $\xi^n + \lambda_n$ for some constant λ so that the two norms are the same. But we will respect more structure: We think of ξ as an indeterminate and consider the polynomial ring $F[\xi]$ over $\mathbb{R}$ or $\mathbb{C}$ (more generally, if the dimension of $\mathcal{H}$ is d, we consider $F[\xi_1, \ldots, \xi_d]$ where the ξ_k's are probabilistically independent and correspond to a basis of $\mathcal{H}$) and perform an orthogonal decomposition of $L^2(\Omega, \sigma(\xi), P)$ which matches the direct sum $\bigoplus_n \mathcal{H}^{\odot n}$. Some comments:

- Typically, $F[t] = \bigoplus_k \mathcal{F}_k$ where $\mathcal{F}_k$ is the set of monomials of degree k. In the case of $L^2(\Omega, \sigma(\xi), P)$ we will instead construct a grading which also respects the L^2 inner product, i.e., we will require that $\langle P_k(\xi), P_\ell(\xi) \rangle = C\delta_{k\ell}$, where e.g., $P_k(\xi)$ represents $\xi^{\odot k}$. Note that as a trivial consequence, we would have that

$$\|\xi^{\odot k} + \xi^{\odot \ell}\|^2 = \|\xi^{\odot k}\|^2 + \|\xi^{\odot \ell}\|^2.$$

- We can then generalize to the case $d > 1$ in a straightforward manner, except now the relevant $\mathcal{F}_k$ consist of terms like $P_{\alpha_1} \ldots P_{\alpha_n}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$ (just as in the polynomial ring case). To check this it is sufficient to verify for $\{\xi_1, \ldots, \xi_n\}$ independent and $\{\eta_1, \ldots, \eta_m\}$ independent that

$$\|C(P_{\alpha_1}(\xi_1) \ldots P_{\alpha_n}(\xi_n)) + C'(P_{\beta_1}(\eta_1) \ldots P_{\beta_m}(\eta_m))\|^2 = C^2(\alpha^k)!\sigma_{\xi_k}^2 + (C')^2(\beta^\ell)!\sigma_{\eta_\ell}^2.$$

  (Here e.g., α^k denotes the k^{th} entry in the multi–index α.) Here we are implicitly using the fact that if e.g., $\xi_1, \ldots, \xi_d$ is a basis for the relevant Gaussian Hilbert space then, by definition, any other element can be written as a linear combination of the ξ_k's and hence it is sufficient to consider polynomials in $\xi_1, \ldots, \xi_d$.

- We now begin to carry out the orthogonalization procedure described. It is clear that we should take

$$\mathcal{F}_0 = F, \quad \mathcal{F}_1 = L^2(\Omega, \sigma(\xi), P),$$

where $F = \mathbb{R}$ or $F = \mathbb{C}$ is the field of scalars. It is clear that $\mathcal{F}_0 \perp \mathcal{F}_1$, since we have centered Gaussians. It follows that

$$\mathcal{F}_2 = \{t_2\xi^2 + t_1\xi + t_0 : \langle t_2\xi^2 + t_1\xi + t_0, t_1'\xi + t_0' \rangle = 0, \forall t_1', t_0' \in \mathbb{R}.\}.$$

This leads to the equation

$$(t_0' t_2 + t_1' t_1)\sigma_\xi^2 + t_0' t_0 = 0, \quad \forall t_0', t_1',$$

so independently setting coefficients of t_0', t_1' to zero, we obtain that $t_1 = 0$, and $-1 = \frac{t_2}{t_0}\sigma_\xi^2$, and so

$$\mathcal{F}_2 = \{\lambda \cdot \sigma_\xi^2 \cdot ((\xi/\sigma_\xi)^2 - 1) : \lambda \in F\} := \{\lambda \cdot P_2(\xi) : \lambda \in F\}.$$

(This agrees from our computation from before.) By induction, we have

$$\begin{aligned}\mathcal{F}_n &= \{\lambda P_n(\xi) := t_n \xi^n + \cdots + t_0 : \lambda, t_n, \ldots, t_0 \in F; : \langle P_n(\xi), \xi^\ell \rangle = 0, \forall \ell < n\} \\ &= \{\lambda P_n(\xi) : \langle P_n, P_{n-1}\rangle = 0, \ldots, \langle P_n, P_0 \rangle = 0\} \\ &= \left\{ \lambda \left(\xi^n + \sum_{j=1}^n \alpha_j P_j(\xi) \right) = \lambda \left(\xi^n - \frac{\langle \xi^n, P_\ell(\xi)}{\ell!} P_\ell(\xi) \right) : \lambda \in F \right\}\end{aligned}$$

where the last two identities follow from the inductive hypothesis that $\{P_1(\xi), \ldots, P_{n-1}(\xi)\}$ is an orthonormal basis for $F(\xi, \xi^2, \ldots, \xi^{n-1})$ (that is, polynomials expressions in ξ up to degree $k-1$) and we have used the fact that

$$\langle P_\ell(\xi), P_\ell(\xi)\rangle = \langle \xi^{\odot \ell}, \xi^{\odot \ell}\rangle = \ell!$$

(which also tells us that the P_ℓ's should be normalized by $(\ell!)^{-1/2}$.)

Simple counting of equations also shows that if e.g., n is odd, then $P_n(\xi)$ only contains odd powers: There are $n-1$ equations $\langle P_n(\xi), \xi^\ell\rangle, \ell = 1, \ldots, n-1$ and each equation constrains only coefficients of even powers in $\langle P_n(\xi), \xi^\ell \rangle$ and so if e.g., n is odd, then $n-1$ is even with $\lfloor (n-1)/2 \rfloor + 1$ even numbers in $\{1, \ldots, n-1\}$, leading to $\lfloor (n-1)/2 \rfloor + 1$ *inconsistent* equations (the inconsistency is clear since difference even moments of ξ receive different combinatorial prefactors) for the $(n+1)/2 = \lfloor (n-1)/2 \rfloor + 1$ coefficients of even powers in the polynomial $P_n(\xi)$ (namely $t_n, t_{n-2}, \ldots, t_0$) forcing them all to vanish. On the other hand, there are one more coefficient of odd powers than number of constraining equations, leading to a one–parameter solution (the parameter corresponds to overall scaling of the polynomial).

Thus, we see here that each $\mathcal{F}_k$ is a one dimensional space over F. More generally, if $\dim(\mathcal{H}) = d$, since $\mathcal{F}_k$ is generated by products of the form $P(\xi_{\alpha_1}) \ldots P(\xi_{\alpha_\ell})$ where $|\alpha| = \alpha_1 + \cdots + \alpha_\ell = k$, the dimension of $\mathcal{F}_k$ is equal to the number of multi–indices with degree k in a d–dimensional space, which is $\binom{k+d-1}{d-1}$. (This is clear since we can consider a $k+d-1$ row of objects, $k-1$ of which are chosen to be dividers which specify what each α_i should be equal to).

- We will now obtain more detailed information on the structure of the decomposition. First note that it is clearly sufficient to consider variables ξ with $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 = 1$, since then given an arbitrary η, η/σ_η satisfies these normalizations and so if $(\eta/\sigma_\eta)^{\odot n} \approx P_n(\eta/\sigma_\eta)$, then by multi–linearity,

$$\eta^{\odot n} \approx \sigma_\eta^n P_n(\eta/\sigma_\eta).$$

The moments of the normalized variable are

$$\mathbf{E}\xi^n = 0, \quad n \text{ odd}; \quad \mathbf{E}\xi^n = (n-1)!! \quad n \text{ even}.$$

Since coefficients of the polynomials P_k clearly involve (even) moments of ξ, it is convenient to return to Feynman diagrams. Let us write

$$P_n(\xi) = \sum_{\gamma_n} C_n(\gamma_n) V(\gamma_n)$$

for some function C_n, where we recall γ_n denotes a Feynman diagram on n vertices and $V(\gamma_n) = 1^{e(\gamma_n)} \cdot \xi^{v(\gamma_n)}$ by our normalization, where $v(\gamma)$ and $e(\gamma)$ denote the number of isolated vertices and edges of of γ_n, respectively. Notice the "parity" of the right hand side: Feynman diagrams contracts via disjoint pairwise expectations and hence the right hand side only contains ξ raised to the powers $n, n-2, \ldots, 0$, but this is consistent with our observation earlier about the parity of $P_n(\xi)$ so we are justified in representing $P_n(\xi)$ using Feynman diagrams.

In any case, we have

$$\begin{aligned}\langle P_n(\xi), \xi^k \rangle &= \sum_{\gamma_n} C_n(\gamma_n) \cdot 1^{e(\gamma_n)} \cdot \mathbf{E}(\xi^{v(\gamma_n)+k}) \\ &= \sum_{\gamma_n} C_n(\gamma_n) \cdot 1^{e(\gamma_n)} \cdot \left(\sum_{\bar{\gamma}_{v(\gamma_n)+k} \supseteq \gamma_n} 1^{e(\bar{\gamma}_{v(\gamma_n)+k})} \right).\end{aligned}$$

Here $\bar{\gamma}$ denotes the fact that we are summing over *complete* Feynman diagrams. We can note immediately that since $v(\gamma_n) + k$ must be even, n must have the same parity as k. Now if we interchange the sum, then we will sum over complete Feynman diagrams on $n + k$ vertices, and given each $\bar{\gamma}_{n+k}$ with ℓ edges connecting the "first" n vertices, the innermost sum would be the sum over all γ_n with $\leq \ell$ edges connecting the relevant vertices prescribed by $\bar{\gamma}_{n+k}$. More precisely, we can rewrite the above as

$$\langle P_n(\xi), \xi^k \rangle = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \sum_{\bar{\gamma}_{n+k} : e(\bar{\gamma}_{n+k})|_{[n]} = \ell} \sum_{\gamma_n : \gamma_n \subset \bar{\gamma}_{n+k}; e(\gamma_n) \leq \ell} C_n(\gamma_n).$$

If we set

$$C_n(\gamma_n) = (-1)^{e(\gamma_n)}$$

then the inner most sum in the last display becomes $(1 + (-1))^\ell$ which is 0 unless $\ell = 0$, corresponding to $k = n$ and the sum reducing to be over perfect matchings between the first n and last n indices, and so we obtain in particular that

$$\langle P_n(\xi), \xi^n \rangle = n! = \langle \xi^n, \xi^n \rangle.$$

We conclude that

$$P_n(\xi) = \sum_{\gamma_n} (-1)^{e(\gamma_n)} V(\gamma_n).$$

Recalling that $\mathcal{F}_n$ is one–dimensional, we have uniquely determined $P_n(\xi)$.

Let us now tabulate some simple extensions:

- The computation of $\langle P_n(\xi), \xi^k \rangle$ can be extended to a formula for computing inner products of the form $\langle \xi_1 \odot \cdots \odot \xi_n, \eta_1 \odot \cdots \odot \eta_m \rangle$: This is 0 by construction if $m \neq n$, and reducing to summing perfect matching as before, we get

  $$\langle \xi_1 \odot \cdots \odot \xi_n, \eta_1 \odot \cdots \odot \eta_m \rangle = \sum_{\sigma \in S_n} \prod_{k=1}^{n} \langle \xi_k, \eta_{\sigma(k)} \rangle.$$

  Further generalization to expressions of the form $\mathbf{E}(X_1 \ldots X_k)$ where $X_i = \xi_{i1} \odot \cdots \odot \xi_{i\ell_i}$ is straightforward: We sum over Feynman diagrams not connecting two vertices which are both part of the same X_i.

- Since an element is zero if and only if the norm is zero, we can also reverse the logic and write e.g.,

$$X_1 \dots X_n = \sum_{\hat{\gamma}} \odot(\hat{\gamma}),$$

with the same restriction over the set of Feynman diagrams being summed, i.e., no edge connecting vertices in the same X_i (since these terms would contribute zero to the relevant expectation). Here $\odot(\hat{\gamma})$ means that $\odot$ replaces ordinary product, so e.g., the term

$$\mathbf{E}(\xi_1\eta_1) \cdot (\xi_2 \odot \eta_2)$$

may contribute to the expression for $XY = (\xi_1 \odot \xi_2)(\eta_1 \odot \eta_2)$. In particular,

$$\xi_1 \dots \xi_n = \sum_{\gamma_n} \odot(\gamma_n) \in \mathcal{F}_n \oplus \mathcal{F}_{n-2} \oplus \dots \oplus \mathcal{F}_{\bar{n}},$$

where the sum is over all Feynman diagrams on n vertices and $\bar{n} \equiv n \mod 2 \in \{0, 1\}$ depending on the parity of n. Notice that this is an inversion formula which allows us to write $\xi_1 \dots \xi_n$ in terms of the orthogonal decomposition we have constructed. It is also noted that this line of reasoning also says that we should in general identify X and Y if it is the case that $\mathbf{E}(XZ) = \mathbf{E}(YZ)$ for all Z: Indeed, taking $Z = (\bar{X} - \bar{Y})$ we see that $\mathbf{E}|X - Y|^2 = 0$.

  - If we have a complex Gaussian $\zeta = \xi + i\eta$ with ξ, η i.i.d. (it is easy to see that in this case, ζ is *symmetric*, that is, $\zeta \sim \lambda\zeta$ for any rotation $\lambda = e^{i\theta}$) then $\zeta^{\odot n} = \zeta^n$, since here $\mathbf{E}\zeta^2 = 0$, so the only term contributing to $P_n(\zeta)$ would be the empty Feynman diagram, corresponding to ζ^n. Of course, this means that $\mathbf{E}\zeta^k = 0$ for all k. It is important here that we use as the underlying inner product $\langle \zeta, \zeta \rangle = \mathbf{E}\zeta^2$ (which in this case is degenerate) and not $\langle \zeta, \zeta \rangle = \mathbf{E}|\zeta|^2$.

- The polynomials $P_n(\xi)$ are the well–known *Hermite polynomials* and we will denote them by H_n from now on. The first few terms are

$$H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3, \dots$$

We can now list a few properties of them:

  - If n is even then $H_n(x)$ only contains even powers and similarly if n is odd.
  - By construction, these are orthogonal polynomials with respect to the standard Gaussian measure $\mathcal{N}(0, 1)$ on $\mathbb{R}$:

$$\langle \xi^{\odot n}, \xi^{\odot m} \rangle = \langle H_n(x), H_m(x) \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n(x) H_m(x) \cdot e^{-x^2/2}\, dx = \delta_{mn} n!$$

Here $\xi \sim \mathcal{N}(0, 1)$ so that $\xi \# P = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

  - From the representation of $\xi^{\odot n}$ where $\xi \sim \mathcal{N}(0, 1)$ in terms of Feynman diagrams, we can also write an explicit expression (combinatorial) expression for H_n:

$$\begin{aligned} H_n(\xi) &= \sum_{\gamma_n} (-1)^{e(\gamma_n)}\ \xi^{v(\gamma_n)} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k)!!(-1)^k\ \xi^{n-2k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{2^k (n-2k)! k!}\ \xi^{n-2k} \end{aligned}$$

- From the expression for $\xi^{\odot n}\xi^{\odot m}$ we can also derive a recursion for the Hermite polynomials:

$$\begin{aligned} H_n(\xi)H_m(\xi) &= \xi^{\odot n}\xi^{\odot m} \\ &= \sum_{\hat{\gamma}_{n+m}} \odot(\hat{\gamma}_{m+n}) \\ &= \sum_{r=0}^{m\wedge n} \binom{m}{r}\binom{n}{r} r!\ \xi^{\odot(m+n-2r)} \\ &= \sum_{r=0}^{m\wedge n} \binom{m}{r}\binom{n}{r} r!\ H_{m+n-2r}(\xi), \end{aligned}$$

where the combinatorial expression comes from the restriction on $\hat{\gamma}_{n+m}$. When $m = 1$, we get the recursion

$$H_n(\xi) = H_{n+1}(\xi) + nH_{n-1}(\xi).$$

In particular, from this it easily follows that

$$\mathbf{E}(H_n(\xi)) = 0, \quad \text{for all } n.$$

- We can also define $e^{\odot \xi}$ as a power series, from which will follow the generating function for $H_n(\xi)$. Let $\xi \sim \mathcal{N}(0,\sigma)$, then we have

$$\begin{aligned} e^{\odot \xi} &= \sum_0^\infty \frac{\xi^{\odot n}}{n!} \\ &= \sum_0^\infty \frac{1}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!} (-1)^k\ (\sigma^2)^k \xi^{n-2k} \\ &= \sum_{k=0}^\infty \frac{1}{k!} \left(-\frac{\sigma^2}{2}\right)^k \sum_{n=2k}^\infty \frac{1}{(n-2k)!} \xi^{n-2k} \\ &= e^{-\sigma^2/2} \sum_{\ell=0}^\infty \frac{\xi^\ell}{\ell!} \\ &= e^{-\sigma^2/2+\xi}. \end{aligned}$$

(It is noted that a side benefit of the interchange of the sum is the disappearance of the ambiguity involving the parity of n.) Recalling the scaling $e^{\odot \xi} = \sum \frac{\sigma^n H_n(\xi/\sigma)}{n!}$, we arrive at the generating function

$$e^{tx - x^2/2} = \sum \frac{t^n}{n!} H_n(x).$$

- The decomposition

$$L^2(\Omega, \sigma(\mathcal{H}), P) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$$

is known as the *Wiener chaos decomposition*. Strictly speaking, $\mathcal{F}_n$ should be constructed out of the L^2 closure of the relevant polynomial spaces. One way to check that the righthand side contains all of $L^2(\Omega, \sigma(\mathcal{H}), P)$ is to note that it contains the exponentials $e^{-i\xi t}$ and then show that if $X \perp e^{-i\xi t}$ for every ξ then it is zero, since $\langle X, e^{-i\xi t}\rangle$ is the Fourier transform of the distribution of X. This can be done immediately in the finite dimensional case where $X \sim t_1\xi_1 + \ldots t_n\xi_n$, and in the infinite dimensional case, the result follows from the fact that all finite dimensional projections vanish, i.e., the conditional expectations: $X_n = E(X \mid \mathcal{F}_n) = 0$, for all n, where $\mathcal{F}_n$ is generate by finitely many ξ_i.

Starting with the isometry

$$\Phi : \mathcal{H} \to L^2(\Omega, \sigma(\mathcal{H}), p),$$

we have constructed a (commutative) graded algebra isomorphism:

$$\Psi : \bigoplus_{n=0}^{\infty} H^{\odot n} \longrightarrow \bigoplus_{n=0}^{\infty} \mathcal{F}_n.$$

This direct sum is called the (symmetric) *Fock space* over $\mathcal{H}$. In what follows for simplicity we consider $\xi \sim \mathcal{N}(0,1)$ and suppose $\mathcal{H}$ is 1–dimensional. We have explicitly the isometries

$$\Psi_n : \mathcal{H}^{\odot n} \to \mathcal{F}_n : \xi^{\odot n} \mapsto H_n(\xi) = \sum_{\gamma_n} (-1)^{e(\gamma_n)}\ \xi^{\bullet v(\gamma_n)}$$

(here $\bullet$ denotes multiplication in L^2) and their inverses

$$\Psi_n^{-1} : \mathcal{F}_n \to \mathcal{H}^{\odot n} : \xi^{\bullet n} \mapsto \sum_{\gamma_n} \xi^{\odot v(\gamma_n)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k)!!\ H_{n-2k}(\xi),$$

where the sum is over all Feynman diagrams on n vertices. We note especially the duality between Feynman diagrams and Hermite polynomials: Hermite polynomials represent $\xi^{\odot n}$ in L^2 whereas Feynman diagrams represents ξ^n in $\sum_{n=0}^{\infty} \mathcal{H}^{\odot n}$. We also note that the above can be rephrased as saying that we now have two products on the space of random variables $L^2(\Omega, \sigma(\mathcal{H}), P)$, denoted by $\odot$ and $\bullet$ such that the (linearly extended) transforms given by:

$$\mathscr{F} : \odot \to \bullet : \mathscr{F}(\xi^{\odot n}) = \sum_{\gamma_n} (-1)^{e(\gamma_n)}\ \xi^{\bullet v(\gamma_n)}$$
$$\mathscr{G} : \bullet \to \odot : \mathscr{G}(\xi^{\bullet n}) = \sum_{\gamma_n} \xi^{\odot v(\gamma_n)}$$

are inverses.

Preservation of grading means

$$\Psi_{m+n}(\xi^{\odot m} \odot \xi^{\odot n}) = \pi_{m+n}(\Psi_m(\xi) \bullet \Psi_n(\xi)) = \pi_{m+n}(H_m(\xi) \bullet H_n(\xi)),$$

where π_{m+n} denotes the projection operator onto $\mathcal{F}_{m+n}$, and, inversely,

$$\Psi_{m+n}^{-1}(H_m(\xi) \bullet H_n(\xi)) = \xi^{\odot m} \odot \xi^{\odot n}.$$

Using the representation

$$\xi^{\odot m} \bullet \xi^{\odot n} = H_m(\xi) \bullet H_n(\xi) = \sum_{\hat{\gamma}_{m+n}} \xi^{\odot v(\hat{\gamma}_{m+n})},$$

we arrive at the tautology

$$\begin{aligned}
\Psi_{m+n}(\xi^{\odot m} \odot \xi^{\odot n}) &= \pi_{m+n}(H_m(\xi) \bullet H_n(\xi)) \\
&= \pi_{m+n}(\xi^{\odot m} \bullet \xi^{\odot n}) \\
&= \pi_{m+n}\left(\sum_{\hat{\gamma}_{m+n}} H_{v(\hat{\gamma}_{m+n})}(\xi)\right) \\
&= H_{m+n}(\xi) \\
&= \Psi_{m+n}(\xi^{\odot(m+n)}).
\end{aligned}$$

Operator Product Expansion

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1 Operator Product Expansion

Recall that the Gaussian field is an isometry $\Phi : \mathcal{H} \to L^2(\Omega, P)$ such that the images are centered Gaussians $\mathcal{N}(0, \sigma)$. Our setting is as follows: We will consider a (simply connected) domain $\mathcal{D} \subset \mathbb{C}$ and consider

$$\mathcal{H} = \mathcal{E}(\mathcal{D}) = \overline{C_0^\infty(\mathcal{D})},$$

the closure with respect to the Dirichlet norm of test functions vanishing on $\partial\mathcal{D}$. The Dirichlet inner product as usual is defined as

$$\langle f, g \rangle_{\mathcal{D}} = \int_{\mathcal{D}} \langle \nabla f, \nabla g \rangle \; dA.$$

(We recall that harmonic functions are minimizers of the Dirichlet energy $\|f\|^2_{\mathcal{D}}$.) By definition, Φ being an isometry means that

$$\mathbf{E}(\Phi(f)\Phi(g)) = \langle \Phi(f), \Phi(g) \rangle_{L^2(\Omega,P)} = \langle f, g \rangle_{\mathcal{D}}.$$

We also note that for centered Gaussians $\mathbf{E}(\Phi(f)\Phi(g))$ is the *covariance*:

$$\mathbf{E}(\Phi(f)\Phi(g) = \mathrm{Cov}(\Phi(f)\Phi(g)) = \mathbf{E}(\Phi(f)\Phi(g)) - \mathbf{E}(\Phi(f))\mathbf{E}(\Phi(g)).$$

(When $\Phi(f) = \Phi(g)$, this is the *variance* of $\Phi(f)$, which we sometimes denote $\sigma_{\Phi(f)}$.)

Clearly, we can also take the perspective that $\Phi : \mathcal{E}(\mathcal{D}) \times \Omega \to \mathbb{R}$. This is useful when we wish to consider some notion of convergence of fields. Specifically, we may have discretized domains $\mathcal{D}^\varepsilon$ and discrete fields

$$\Phi^\varepsilon : \mathcal{E}(\mathcal{D}^\varepsilon) \times \Omega \to \mathbb{R}$$

and we must make sense of how Φ^ε converges to Φ. Since a discrete function can be viewed as a sum of delta masses, it may be more natural to consider not a space of functions, but a space of *measure* as the "physical" space $\mathcal{H}$. Indeed, by the Krein–Milman theorem, the space of delta measures is dense in $(C_0)^*$, so we may interpret field convergence to mean that

$$\Phi^\varepsilon(\mu^\varepsilon, \omega^\varepsilon) \to \Phi(\mu, \omega)$$

whenever μ^ε converges to μ in some suitable sense and $(\omega^{\varepsilon_1}, \omega^{\varepsilon_2}, \ldots\ldots, \omega)$ is drawn from the correponding Skorohod coupling measure driving the (weak) convergence of the lattice model under consideration (e.g., the measure on curves generated by percolation converges to SLE_6).

But for now we will take a more down to earth perspective: Convergence of *correlation functions*. To motivate this concept, let us temporarily return to the most naïve interpretation of a random field as a random function, in which case we consider a field to be

$$\Phi^\varepsilon : \mathbb{C}^\varepsilon \times \Omega \to \mathbb{R}.$$

(This can be related to functions by considering $\Phi(z)$ to be the limit of $\Phi(f_{\delta,z})$ where $f_{\delta,z}$ is any test function converging to δ_z.) We may easily make sense of convergence of Φ^ε in e.g., the distributional sense *if* there is some way to make sense of what it means for ω to lie in a compact set. However, since for now there is no underlying geometry on the space Ω, we will integrate out the ω variable: I.e., we will take expectation and require that for all test functions φ,

$$\mathbf{E}\left[\int_{\mathbb{C}} \Phi^\varepsilon(z,\omega)\ \varphi(z)\ dz\right] \longrightarrow \mathbf{E}\left[\int_{\mathbb{C}} \Phi(z,\omega)\ \varphi(z)\ dz\right].$$

Now we observe that by Fubini's Theorem, this is equivalent to

$$\int_{\mathbb{C}} \mathbf{E}(\Phi^\varepsilon(z,\omega))\ \varphi(z)\ dz \longrightarrow \int_{\mathbb{C}} \mathbf{E}(\Phi(z,\omega))\ \varphi(z)\ dz.$$

The function(al)

$$\mathcal{C}_\Phi(z) = \mathbf{E}(\Phi(z,\omega))$$

is exactly the *correlation function*, which clearly can be viewed as a (integral) functional. Clearly, convergence in the sense of the above display means convergence of the relevant first moments.

The previous paragraph must be understood to be entirely formal: First of all, with centered Gaussians, the first moment is identically zero and thus we have to begin with the second moment (and later all even moments). A naïve computation of second moment immediately yields infinity: Let us now take the representation that $\Phi \sim \xi^\alpha e_\alpha$, where $\{e_\alpha\}_\alpha$ is an orthonormal basis for $\mathcal{H}$ and we have used Einstein summation notation (so that $\Phi(f) = \langle X, f\rangle = \sum_\alpha f^\alpha \xi^\alpha$). It is immediate then that

$$\mathbf{E}(|\Phi|^2) = \sum_\alpha \sigma^2_{\xi^\alpha},$$

which is typically infinity. More precisely, we will shortly see that the correlation at two points diverges like Green's function. Finally, the *operator product expansion* gives asymptotic expansion of the the correlation function (and hence of the field) as one point approaches the other.

- The *n–point correlation function* of Φ is a continuous assignment

  $$(z_1,\dots,z_n) \mapsto \mathcal{C}_\Phi(z_1,\dots,z_n),$$

  such that

  $$\mathbf{E}(\Phi(f_1)\dots\Phi(f_n)) = \int f_1(z_1)\dots f_n(z_n)\ \mathcal{C}_\Phi(z_1,\dots,z_n)\ dz_1\dots dz_n.$$

  (Here, by abuse of notation, $dz = dx \wedge dy$. We shall be more careful later.) In particular, by Wick's formula,

  $$\mathcal{C}_\Phi(z_1,\dots,z_n) = \sum \prod_k \mathcal{C}_\Phi(z_{i_k}, z_{j_k}).$$

  By very twisted interpretation we can think of $\mathcal{C}_\Phi$ as a multi–linear form, but for now it is just a function satisfying the above recursion.

- Let us now transfer the structure of the chaos decomposition to the computation of correlation functions. Let $f, g \in \mathcal{H}$, then by notation and construction we have the tautology that

$$(\Phi \odot \Phi)(f,g) = \Phi(f \odot g) = \Phi(f) \odot \Phi(g) = \Phi(fg).$$

(On the function space $\mathcal{H}$, $f \odot g$ corresponds to pointwise multiplication of functions.) Therefore we have on the one hand

$$\mathbf{E}[(\Phi \odot \Phi)(f_1, f_2) \bullet (\Phi \odot \Phi)(g_1, g_2)] = \int (f_1 f_2)(z_f)(g_1 g_2)(z_g)\ \mathcal{C}_{\Phi\odot\Phi}(z_f, z_g)\ dz_f dz_g,$$

and on the other hand, by Wick's formula,

$$\begin{aligned}
&\mathbf{E}[(\Phi(f_1) \odot \Phi(f_2)) \bullet (\Phi(g_1) \odot \Phi(g_2))] \\
&= \mathbf{E}[\Phi(f_1)\Phi(g_1)] \cdot \mathbf{E}[\Phi(f_2)\Phi(g_2)] + \mathbf{E}[(\Phi(f_1)\Phi(g_2)] \cdot \mathbf{E}[(\Phi(f_2)\Phi(g_1)] \\
&= \left(\int f_1(z_{f_1}) g_1(z_{g_1})\ \mathcal{C}_\Phi(z_{f_1}, z_{g_1})\ dz_{f_1} dz_{g_1} \right) \cdot \left(\int f_2(z_{f_2}) g_2(z_{g_2})\ \mathcal{C}_\Phi(z_{f_2}, z_{g_2})\ dz_{f_2} dz_{g_2} \right) \\
&+ \left(\int f_1(z_{f_1}) g_2(z_{g_2})\ \mathcal{C}_\Phi(z_{f_1}, z_{g_2})\ dz_{f_1} dz_{g_2} \right) \cdot \left(\int f_2(z_{f_2}) g_1(z_{g_1})\ \mathcal{C}_\Phi(z_{f_2}, z_{g_1})\ dz_{f_2} dz_{g_1} \right).
\end{aligned}$$

Iterating the integrals and comparison with the previous display yields that

$$\begin{aligned}
\mathcal{C}_{\Phi\odot\Phi}(z_f, z_g) &= \frac{1}{2}[\mathcal{C}_\Phi(z_{f_1}, z_{g_1}) \cdot \mathcal{C}_\Phi(z_{f_2}, z_{g_2}) + \mathcal{C}_\Phi(z_{f_1}, z_{g_2}) \cdot \mathcal{C}_\Phi(z_{f_2}, z_{g_1})]_{z_{f_1} = z_{f_2} = z_f, z_{g_1} = z_{g_2} = z_g} \\
&= (\mathcal{C}_\Phi(z_f, z_g))^2.
\end{aligned}$$

From this we conclude that 1) correlation *functionals* of Wick products should be computed according to Wick's formula (i.e., sum (the values) over the relevant Feynman diagrams) and that 2) the correlation *functions* themselves satisfy $\mathcal{C}_{\Phi\odot\Psi} = \mathcal{C}_\Phi \cdot \mathcal{C}_\Psi$. In the context of field theory, this is how we understand Wick's multiplication $\odot$: The product is exactly constructed to replicate point–wise multiplication of functions.

- By taking an approximation to the identity and integrating by parts, we can also make the concept of correlation functions more literal: Indeed, suppose

$$f_{\varepsilon,z} \to \delta_z, \quad \text{as } \varepsilon \to 0; \quad \text{supp}(f_{\varepsilon,z}) \subset B_\varepsilon(z),$$

then by the Lebesgue differentiation theorem,

$$\mathbf{E}[\Phi(f_{\varepsilon,z_1})\Phi(f_{\varepsilon,z_2})] = \int\int \mathcal{C}_\Phi(\zeta, \eta)\ f_{\varepsilon,z_1}(\zeta) f_{\varepsilon,z_2}(\eta)\ d\zeta d\eta \to \mathcal{C}_\Phi(z_1, z_2).$$

That is, we may think of the correlation *function* as representing the expectation of Φ applied at delta functions and in this way we may pass from the abstract definition of Φ as an isometry between function spaces to its representation as a genuine spatial function represented by (and often identified with) $\mathcal{C}_\Phi$.

- We would now like to define a notion of derivatives of fields. We will sometimes parametrize $\mathbb{C}$ by the variables $z, \bar{z}$ (other times, when it is clear, dz denotes the usual $2d$ integral) and use the notation

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

so that

i) $\partial\bar{\partial} = \bar{\partial}\partial = \Delta$;

ii) if $f = u + iv$, then $\bar{\partial}\bar{f} = \overline{\partial f}$.

ii) a *holomorphic function* f satisfies $\bar{\partial} f = 0$;

iii) if u is *harmonic*, then ∂u is holomorphic.

It is also readily verified that if $dx \wedge dy$ is the volume form in $2d$, then

$$dz \wedge d\bar{z} = -i(dx \wedge dy).$$

Also, using Stoke's theorem

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega,$$

(where ω is a k–form and d represents exterior differentiation) we have that if f is holomorphic, then

$$\int_{\partial\Omega} f(z)\ dz = \int_\Omega d(f dz) = \int_\Omega \frac{\partial f}{\partial z}\ (dz \wedge dz) + \frac{\partial f}{\partial \bar{z}}\ (d\bar{z} \wedge dz) = 0.$$

(This is *Cauchy's Theorem.*)

From the perspective of correlation functions and distributions, it is natural to define the derivatives by the equation

$$\mathcal{C}_{\partial^\alpha \bar{\partial}^\beta \Phi} = \partial^\alpha \bar{\partial}^\beta \mathcal{C}_\Phi,$$

where α, β are multi–indices (we have suppressed explicit reference to how many parameters e.g., $\mathcal{C}_\Phi$ takes). That is, $\partial^\alpha \bar{\partial}^\beta$ is defined to be the field with the correlation function as displayed. We may wonder whether $\partial^\alpha \bar{\partial}^\beta$ lies in the Fock space as described before. One way to think about this is to recall that Wick's product of fields corresponds to products of correlation functions, and thus, since the Fock space contains the closure of of all possible (Wick) polynomials of Φ we can conclude that $\partial^\alpha \bar{\partial}^\beta \Phi$ lies in the Fock space if of $\partial^\alpha \bar{\partial}^\beta \mathcal{C}_\Phi$ can be expressed as a limit of polynomials in $\mathcal{C}_\Phi$.

It is immediately clear from this definition (via correlation functions) that Leibnitz rule for Wick products follow immediately from the usual Leibnitz rule for functions: If X, Y are fields, then e.g.,

$$\partial(X \odot Y) = (\partial X) \odot Y + X \odot (\partial Y).$$

Finally, we also say that a X is a *holomorphic field* if its correlation function is holomorphic *away from singularity* (the last point will become clear once we explicitly compute the 2–point correlation function $\mathcal{C}_\Phi$).

- To derive an expression for $\mathcal{C}_\Phi(z_f, z_g)$ we return to the definition of the Dirichlet norm. Recall that we have for $\langle f, g\rangle_{\mathcal{E}(\mathcal{D})} = \langle \nabla f, \nabla g\rangle$ (here $\langle , \rangle$ will denote either the Euclidean inner product or the usual L^2 inner product of functions).

  - Let us first observe that the Dirichlet inner product is invariant under conformal maps: Suppose $\varphi : \mathcal{D}' \to \mathcal{D}$ is a conformal map, then

$$\int_{\mathcal{D}'} \langle \nabla f(z'), \nabla g(z')\rangle\ dz' = \int_{\mathcal{D}} \langle \nabla f(\varphi(z')), \nabla g(\varphi(z'))\rangle \cdot |D\varphi^{-1}|\ dz$$

by the change of variables formula, where $|D\varphi|$ is the determinant of the Jacobian of φ. The Cauchy–Riemann equations imply the identity that

$$[D\varphi][D\varphi]^T = |D\varphi|,$$

where $[D\varphi]$ now denotes the Jacobian matrix of φ. By the chain rule, we have e.g.,

$$\nabla f(\varphi(z')) = [D\varphi(z')] \cdot [\nabla f(\varphi(z')] = [D\varphi(z')] \cdot [\nabla f(z)],$$

therefore (here $[\cdot] \cdot [\cdot]$ denotes matrix multiplication)

$$\langle \nabla f(\varphi(z')), \nabla g(\varphi(z')\rangle = \langle [D\varphi]\cdot[\nabla f], [D\varphi]\cdot[\nabla g]\rangle = \langle [D\varphi]\cdot[D\varphi]^T\cdot[\nabla f], \nabla g\rangle = |D\varphi|\langle \nabla f, \nabla g\rangle.$$

Since $|D\varphi^{-1}| = |D\varphi|^{-1}$, we conclude that

$$\langle f, g\rangle_{\varphi(\mathcal{D})} = \langle f \circ \varphi, g \circ \varphi\rangle_{\mathcal{D}}.$$

- Next we represent $\langle \nabla f, \nabla g\rangle$ in terms of the Laplacian. Formally, integration by parts gives that

$$\langle \nabla f, \nabla g\rangle = \langle f, -\Delta g\rangle = \langle (-\Delta)^{1/2} f, (-\Delta)^{1/2} g\rangle.$$

The first equality in the above gives that $-\Delta$ is a positive operator, so the second equality can be understood e.g., in terms of eigenvalues and eigenvectors of the Laplacian: More precisely, let $\{e_j\}_j$ be eigenvectors of $(-\Delta)$ which form an orthonormal basis under the L^2 inner product and let $\{\lambda_j\}_j$ be the associated eigenvalues. Then writing $f = \alpha^j e_j, g = \beta^j e_j$, we obtain that

$$\langle f, -\Delta g\rangle = \langle \alpha^j e_j, \lambda_j^{-1} \beta^j e_j\rangle = \langle \lambda_j^{1/2} \alpha^j e_j, \lambda_j^{1/2} \beta^j e_j\rangle = \langle (-\Delta)^{1/2} f, (-\Delta)^{1/2} g\rangle.$$

In particular, $\{\lambda_j^{-1/2} e_j\}_j$ is an orthonormal basis under the Dirichlet inner product. Said differently, the space $\mathcal{E}(\mathcal{D})$ (with the Dirichlet inner product) can be represented as $(-\Delta)^{-1/2} L^2(\mathcal{D})$ (with the L^2 inner product).

- It is now straightforward to compute the correlation function. Using the representation as described above, we can write $f = (-\Delta)^{-1/2} \tilde{f}, g = (-\Delta)^{-1/2} \tilde{g}$, so that we have

$$\mathbf{E}[\Phi(\tilde{f})\Phi(\tilde{g})] = \langle \tilde{f}, \tilde{g}\rangle = \langle (-\Delta)^{-1} \tilde{f}, \tilde{g}\rangle.$$

Now let $G(z_g, z_f)$ denote the Dirichlet Green's function, that is, G satisfies

$$-\Delta G(z_f, z_g) = \delta_{z_g}, \quad G\mid_{\partial\mathcal{D}}= 0,$$

so that

$$\int_{\mathcal{D}} G(z_g, z_f)\tilde{f}(z_f) = ((-\Delta)^{-1}\tilde{f})(z_g).$$

We conclude that $\mathbf{E}(\Phi(\tilde{f})\Phi(\tilde{g})] = \int_{\mathcal{D}} G(z_g, z_f)\ \tilde{f}(z_f)\tilde{g}(z_g)\ dz_f dz_g$. Comparing with the definition of the correlation function, we conclude that

$$\mathcal{C}_\Phi(z_f, z_g) = G(z_f, z_g).$$

Notice that we clearly have a singularity as $z_f \to z_g$, and since by Wick's formula, all n–point correlation functions can be expressed in terms of the 2–point correlation function, it is (in principle) sufficient to study the asymptotic expansion for Green's function, as z_f tends to z_g.

- We recall that Green's function is conformally invariance, so in particular, if

$$\varphi : D \to \mathbb{D} : \zeta \mapsto 0, \varphi'(\zeta) > 0$$

is the (normalized at ζ) uniformization map, then

$$G_D(\zeta, z) = G_{\mathbb{D}}(\varphi(\zeta), \varphi(z)) = -\log|\varphi(\zeta) - \varphi(z)|.$$

Therefore, the function $u(\zeta, z) = G_D(\zeta, z) + \log|\zeta - z|$ satisfies

$$u(\zeta, \zeta) = \log(1/\varphi'(\zeta)) := c(\zeta),$$

where $1/\varphi'(\zeta)$ is the *conformal radius*. Indeed,

$$u(\zeta, z) = -\log|\varphi(\zeta) - \varphi(z)| + \log|\zeta - z| = -\log\frac{|\varphi(\zeta) - \varphi(z)|}{|\zeta - z|},$$

from which the conclusion is immediate by taking $z \to \zeta$. (We have subtracted the singularity of Green's function at ζ, up to the uniformization map.)

We can now perform expansion for $u(\zeta, z)$. We have

$$\begin{aligned} u(\zeta, z) &= -\log\left|\frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}\right| \\ &= -\log\left|\varphi'(\zeta) + \frac{\varphi''(\zeta)}{2}(z - \zeta) + \frac{\varphi'''(\zeta)}{6}(z - \zeta)^2 + O((z - \zeta))^3\right| \\ &= -\log\varphi'(\zeta) + \frac{1}{\varphi'(\zeta)} \cdot \operatorname{Re}\left[\frac{\varphi''}{2}(z - \zeta) + \frac{\varphi'''}{6}(z - \zeta)^2 + O((z - \zeta)^3)\right] \\ &\qquad - \frac{1}{2}\frac{1}{(\varphi'(\zeta))^2} \cdot \operatorname{Re}\left[\frac{(\varphi'')^2}{4}(z - \zeta)^2 + O((z - \zeta)^3)\right]. \end{aligned}$$

Here we have used that

$$\log|z + \varepsilon| = \operatorname{Re}(\log(z + \varepsilon)) = \log|z| + \operatorname{Re}\left[\frac{1}{z} \cdot \varepsilon - \frac{1}{2}\frac{1}{z^2} \cdot \varepsilon^2 + O(\varepsilon^3)\right]$$

and the normalization $\varphi'(\zeta) > 0$. (Alternatively, we can write $\log|z| = \log(z\bar{z})^{1/2}$ and expand in z *and* $\bar{z}$.)

  - We immediately obtain from this expansion that

$$\begin{aligned} \mathcal{C}_\Phi(\zeta, z) = \log_D(\zeta, z) &= u(\zeta, z) - \log|\zeta - z| \\ &= c(\zeta) + \log\frac{1}{|\zeta - z|} + o(1), \end{aligned}$$

    where $o(1) \to 0$ as $z \to \zeta$.
  - We recall that the *Schwarzian derivative* of φ is defined as

$$S_\varphi = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 = N'_\varphi - \frac{N_\varphi^2}{2},$$

where $N_\varphi = (\log \varphi')'$ is the *pre–Schwarzian derivative*. (The Schwarzian derivative vanishes if and only if φ is a fractional linear transformation.) It is easy to check that we can also write

$$u(\zeta, z) = c(\zeta) + \mathrm{Re}\left[\frac{1}{2}N_\varphi(\zeta)\cdot(z-\zeta) + \frac{1}{6}\left[S_\varphi(\zeta) + \frac{3}{4}N_\varphi^2(\zeta)\right]\cdot(z-\zeta)^2 + O((z-\zeta)^3)\right].$$

On the other hand, the *Schwarzian* of D is defined as ($\log|z| = \log((z\bar{z})^{1/2})$)

$$2S(\zeta, z) = -\partial_\zeta \partial_z u(\zeta, z) = \frac{1}{2}\cdot\left[\frac{1}{(z-\zeta)^2} - \frac{\varphi'(z)\varphi'(\zeta)}{(\varphi(z)-\varphi(\zeta))^2}\right]$$

- We have by Wick's formula that e.g.,

$$\Phi(f) \odot \Phi(g) = \Phi(f)\Phi(g) - \mathbf{E}[\Phi(f)\Phi(g)].$$

We have already computed the correlation function of $\Phi(f) \odot \Phi(g)$ and noted the singularity in the expression $\mathbf{E}[\Phi(f)\Phi(g)]$ along the "diagonal" and performed the relevant asymptotic expansion. Understanding the above in the sense of correlation functions (recall we have also already computed the correlation function for $\Phi^{\odot 2}$) we may replace f, g by spatial parameters ζ, z, and arrive at the *operator product expansion*

$$\Phi(\zeta)\Phi(z) = \log\frac{1}{|\zeta - z|} + c(\zeta) + \Phi^{\odot 2}(\zeta) + o(1),$$

where $o(1) \to 0$ as $z \to \zeta$.

By Wick's formula, this procedure can be generalized. In case we have a holomorphic field, we can write a Laurent series

$$X(\zeta)Y(z) = X(\zeta) \odot Y(z) + \mathbf{E}[X(\zeta)Y(z)] = \sum C_n(z)(\zeta - z)^n, \quad \zeta \to z.$$

Clearly, the singular part comes from terms of the form $\mathbf{E}(X(\zeta)Y(z))$ (they are mostly polynomials of Green's functions), and the non–singular parts consist of Wick's product of fields and their derivatives. We also use the notation

$$X * Y = C_0, \quad X *_n Y = C_n, \quad n > 0 \quad \text{and} \quad X(\zeta)Y(z) \sim \sum_{n<0} C_n(z)(\zeta - z)^n.$$

$X * Y$ is called the *OPE product* of X and Y. We note that the singular part is preserved under differentiation, e.g.,

$$J(\zeta)J(z) \sim -\frac{1}{(\zeta - z)^2}, \quad J(\zeta)\bar{J}(z) \sim -\bar{\partial}\left(\frac{1}{\zeta - z}\right) = 0.$$

Let us list some properties and examples:

  - OPE coefficients are combinations of Wick products so are clearly Fock space fields.
  - OPE multiplication is neither associative nor commutative.
  - By differentiating the Laurent series, it can be shown that Leibnitz rule is satisfied:

$$\partial(X *_n Y) = (\partial X *_n Y) + (X *_n \partial Y).$$

- Recall that by construction (assuming that $\Phi \sim \mathcal{N}(0,1)$) $\Phi^{\odot n} = H_n(\Phi)$. Now let us observe
$$\mathbf{E}[\Phi^{*2}] = \mathbf{E}[\Phi^{\odot 2} + c] = c,$$
and it in fact can be checked (e.g., by recursion) that
$$\Phi^{\odot n} = c^{n/2} H_n^*(\Phi/\sqrt{c}),$$
so c scales like the variance.
- From the previous item we now have also that
$$e^{*\alpha\Phi} = e^{c\alpha^2/2} e^{\odot \alpha\Phi}.$$
- The vertex fields are denoted $\mathcal{V}^\alpha = e^{*\alpha\Phi}$. They have the OPE
$$\mathcal{V}^\alpha(\zeta)\mathcal{V}^\alpha(z) = \frac{1}{|\zeta - z|^{\alpha\beta}} \cdot \sum_{j,k=0}^{\infty} (\zeta - z)^j (\bar{\zeta} - \bar{z})^k.$$
- The *stress energy tensor* is defined as $T = -\frac{1}{2}(J * J)$. Since
$$\mathbf{E}[J(\zeta)J(z)] = \partial_\zeta \partial_z G = -\frac{1}{2}\frac{1}{(\zeta - z)^2} + \partial_\zeta \partial_z u(\zeta, z) = -\frac{1}{2}\frac{1}{(\zeta - z)^2} - 2S(\zeta, z),$$
we see that
$$T = -\frac{1}{2} J \odot J + S.$$
It can also be shown that T satisfies the OPE
$$T(\zeta)T(z) \sim \frac{1/4}{(\zeta - z)^4} + \frac{2T(z)}{(\zeta - z)^2} + \frac{\partial T(z)}{(\zeta - z)}.$$