Notes on Smirnov's Paper

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1 Smirnov's Event [1]

For a triangular lattice with mesh δ, we define an event $Q_\alpha(z)$, where $\alpha \in \{1, \tau, \tau^2\}$, $z \in \omega$ and $\tau \equiv exp(\frac{2\pi i}{3})$, as an occurrence of a *simple* path going from the arc $a(\alpha)a(\tau\alpha)$ to the arc $a(\tau^2)a(\alpha)$, and separating z from the arc $a(\tau\alpha)a(\tau^2\alpha)$ (see figure 1). We let $H_\alpha(z)$ denote the probability of $Q_\alpha(z)$.

2 Remark 6 [1]

Claim: Suppose

$$f(z) = A(z) + \tau B(z) + \tau^2 C(z),$$

where A, B, C are real-valued functions. Then if

$$f'(z) = A_x + \tau B_x + \tau^2 C_x = \tau^2 A_\tau + B_\tau + \tau C_\tau,$$

then f satisfies the Cauchy-Riemann equations (and hence is analytic) if

$$A_x = B_\tau, B_x = C_\tau, C_x = A_\tau,$$

where A_x denotes the partial derivative in the x direction and A_τ denotes

$$\lim_{\epsilon \to 0} \frac{f(z + \epsilon\tau) - f(z)}{\epsilon\tau}.$$

Proof: This is a straightforward computation.

3 Cauchy-Riemann Equations, Harmonicity and Cauchy's Theorem

If $f(z) = u(z) + iv(z)$ (u, v real) is analytic, then necessarily

$$\begin{aligned} & f'(z) \\ = & \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ = & -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

This can easily be seen by taking real and then purely imaginary values for h in the definition of the derivative and setting them equal. This gives the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Given this and using existence and equality of mixed partials, we see that

u and v are *harmonic* ,

i.e., $\triangle u = u_{xx} + u_{yy} = 0$ and similarly for v. We say that v is the *conjugate harmonic function* of u.

Conversely, $f(z) = u(z) + iv(z)$ is analytic if u and v are conjugate harmonic functions satisfying the Cauchy-Riemann equations.

Green's Theorem states that

$$\int_{\partial D} f(x,y)dx + g(x,y)dy = \int\int_D \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dxdy,$$

where D is a region in the plane with boundary ∂D.

Now if f is analytic and hence satisfies the Cauchy-Riemann equations, then for D contained in the region of analyticity, we have by Green's Theorem

$$\begin{aligned} \int_{\partial D} f(z)dz &= \int_{\partial D} f(z)dx + if(z)dy \\ &= \int\int_D \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} dxdy \\ &= 0. \end{aligned}$$

This is exactly *Cauchy's Theorem.*

4 Neumann Condition (Equation (13) in [1])

Claim: Suppose A, B and C are real-valued functions such that

$$A + \frac{i}{\sqrt{3}}(B - C), \tag{4.1}$$ anal1

$$B + \frac{i}{\sqrt{3}}(C - A), \tag{4.2}$$ anal2

and

$$C + \frac{i}{\sqrt{3}}(A - B) \tag{4.3}$$ anal3

are analytic, then

$$A_\eta = B_{\tau\eta}, \tag{4.4}$$ conc

where X_x denotes the partial derivative in the x–direction of the function X, τ is a unit vector pointing at $\frac{2\pi i}{3}$ and $\tau\eta$ denotes the vector η rotated by $\frac{2\pi}{3}$. By the symmetry of equations (4.1), (4.2) and (4.3) we also get two other such differential conditions.

Proof: Cauchy–Riemann equations imply

$$A_x = \frac{1}{\sqrt{3}}(B_y - C_y), \quad A_y = \frac{1}{\sqrt{3}}(C_x - B_x); \tag{4.5}$$ cauch1

$$B_x = \frac{1}{\sqrt{3}}(C_y - A_y), \quad B_y = \frac{1}{\sqrt{3}}(A_x - C_x); \tag{4.6}$$ cauch2

$$C_x = \frac{1}{\sqrt{3}}(A_y - B_y), \quad C_y = \frac{1}{\sqrt{3}}(B_x - A_x). \tag{4.7}$$ cauch3

Next let $\eta = (\eta_1, \eta_2)^T$ be any unit vector, then

$$\begin{aligned} A_\eta &= \eta_1 A_x + \eta_2 A_y \\ &= \frac{\eta_1}{\sqrt{3}}\left(B_y - \frac{1}{\sqrt{3}}(B_x - A_x)\right) + \frac{\eta_2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(A_y - B_y) - B_x\right) \end{aligned}$$

by (4.5) and (4.7). Regrouping terms in the last line, we get

$$A_\eta = \frac{1}{3}(\eta_1 A_x + \eta_2 A_y) + \left(\frac{\eta_1}{\sqrt{3}} - \frac{\eta_2}{3}\right) B_y + \left(-\frac{\eta_1}{3} - \frac{\eta_2}{\sqrt{3}}\right) B_x.$$

So we conclude

$$\left(\frac{\eta_1}{\sqrt{3}} - \frac{\eta_2}{3}\right) B_y + \left(-\frac{\eta_1}{3} - \frac{\eta_2}{\sqrt{3}}\right) B_x = \frac{2}{3}(\eta_1 A_x + \eta_2 A_y),$$

i.e.,

$$\left(\frac{\sqrt{3}\eta_1}{2}-\frac{\eta_2}{2}\right)B_y+\left(-\frac{\eta_1}{2}-\frac{\sqrt{3}\eta_2}{2}\right)B_x=\eta_1 A_x+\eta_2 A_y. \tag{4.8}$$ step1

On the other hand, multiplying η by the usual two–dimensional rotation by $\frac{2\pi}{3}$ matrix, we find that

$$\tau\eta=\left(-\frac{\eta_1}{2}-\frac{\sqrt{3}\eta_2}{2},\frac{\sqrt{3}\eta_1}{2}-\frac{\eta_2}{2}\right)^T,$$

so

$$B_{\tau\eta}=\left(-\frac{\eta_1}{2}-\frac{\sqrt{3}\eta_2}{2}\right)B_x+\left(\frac{\sqrt{3}\eta_1}{2}-\frac{\eta_2}{2}\right)B_y.$$

Comparison with (4.8) now gives (4.4). □

References

Smir1 [1] Stanislav Smirnov *Critical Percolation in the Plane. I. Conformal Invariance and Cardy's Formula. II. Continuum Scaling Limit.*