On Convergence to SLE_6

(joint work with I. Binder & L. Chayes)

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Percolation

Regular lattice:

- square
- hex. tiling

color edge/sites: blue w.p. p; yellow w.p. $1-p$

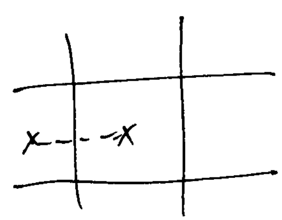
Then $\exists\ 0 < p_c < 1$ (see e.g., Grimmett book).

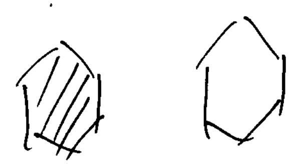
s.t.

- $p > p_c$: $P(0 \rightsquigarrow \infty) > 0$
- $p < p_c$: $P(0 \rightsquigarrow x) \approx e^{-|x|/\xi(p)}$

2D duality

E.g.



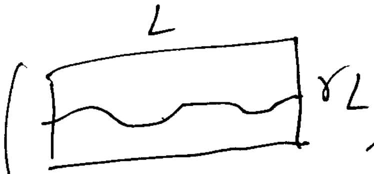


(self-dual).

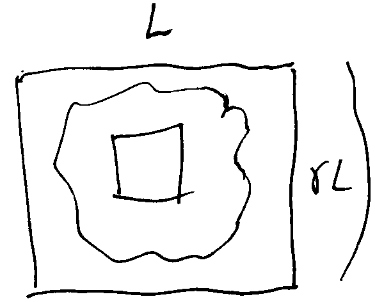
Criticality ($p = p_c$).

- No percolation of either type.

- $C_1 |x-y|^{-\mu_1} \leq P\big($  $\big) \leq C_2 |x-y|^{-\mu_2}$

- $0 < C_1(r) \leq P\big($  $\big) \leq C_2(r) < 1.$

- RSW Estimates:

$0 < C_1(r) \leq P\big($  $\big) \leq C_2(r) < 1.$

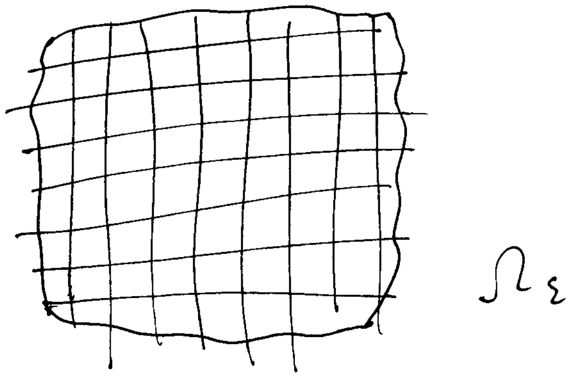
Have:

1). Scale invariance

2). Universal, e.g. independent of lattice ...

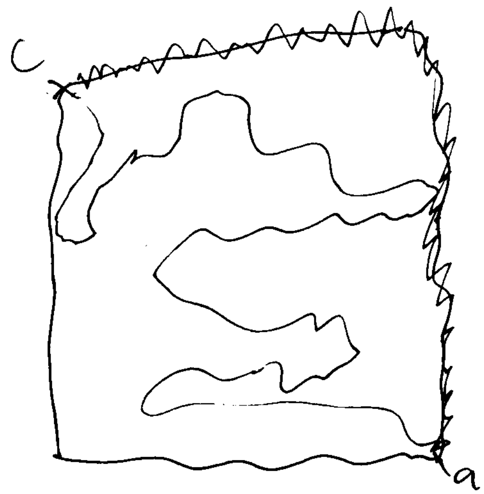
Scaling Limit

At $p = p_c$, study limit



take $\varepsilon \to 0$.

Interface.



Given any blue/yellow config.

$\exists$ interface

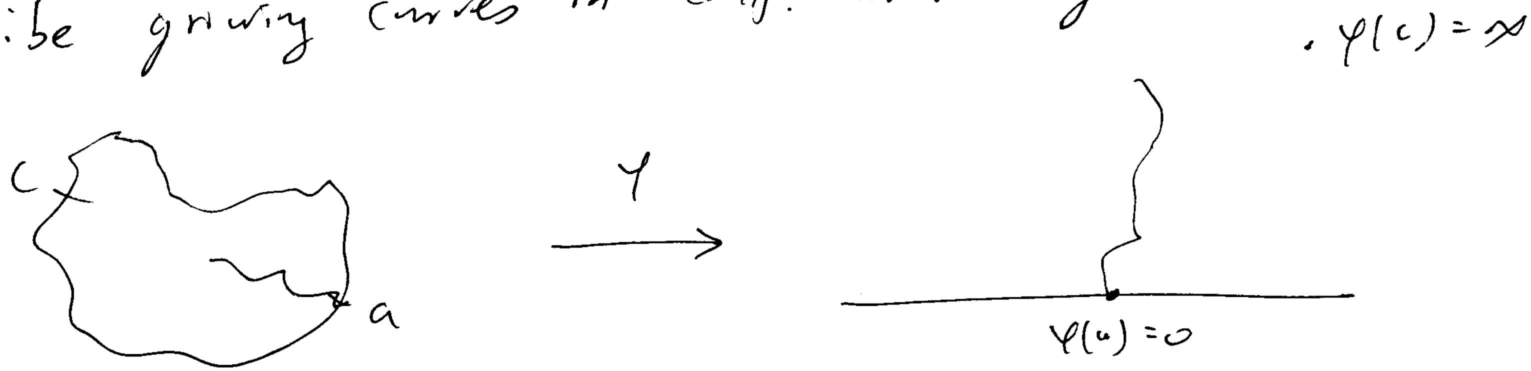
$\rightsquigarrow$ induces measure μ_ε on

$\{$curves $a \rightsquigarrow c\}$

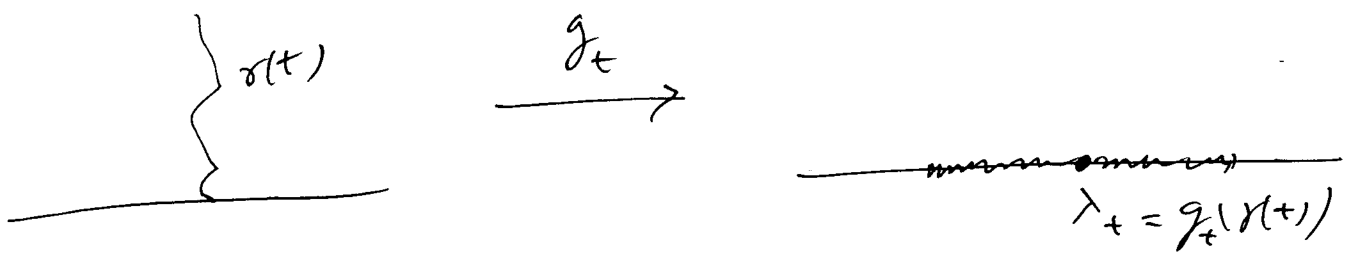
Goal. $\mu_\varepsilon \longrightarrow SLE_6$

SLE.

Describe growing curves in conf. inv. way:



On $\mathbb{H}$:



Löewner ('23)

$$\dot{g}_t = \frac{2}{g_t - \lambda_t}$$

Schramm ('99)

$$\lambda_t = \sqrt{\kappa}\, B(t) \rightsquigarrow \text{(chordal) } SLE_\kappa$$

$B(t)$: Brownian motion

Defining Properties (Schramm's Principle)

(I). Conformal Invariance

$$\varphi : \Omega \longrightarrow \varphi(\Omega)$$

$$\varphi \# \mu (\Omega, a, c) = \mu(\varphi(\Omega), \varphi(a), \varphi(c)).$$

$$(\varphi \# \mu(A) = \mu(\varphi^{-1}(A)))$$

(II). Domain Markov property

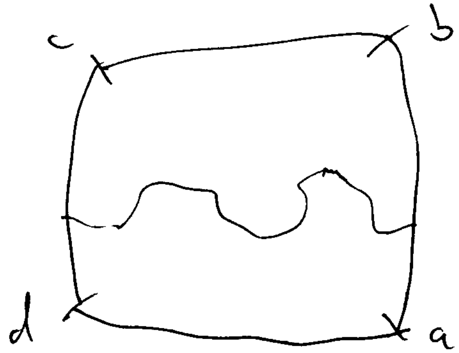
$$\mu(\Omega, a, c)|_{\gamma'} = \mu(\Omega \setminus \gamma', a', c).$$

μ satisfies (I) & (II) $\Longleftrightarrow$ $\mu = SLE_\kappa$, some κ.

Observable.

- Need (I) and (II) from the model
- One observable for ALL domains.

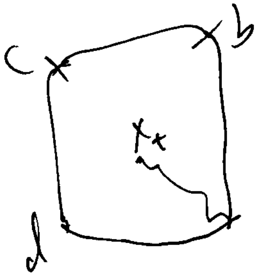
Crossing Probability



$C_\varepsilon(\Omega, a, b, c, d)$

function of domain, etc.

For percolation, easy to see that



$$C_\varepsilon(\Omega, a \mid X^\varepsilon_{[0,t]}) = C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}, X^\varepsilon_t). \qquad (\#)$$

Martingale

I.e., if

$$\mathbb{1}_{\mathcal{C}_\Omega} = \{ \text{indicator of crossing event} \}.$$

then

$$\mathbb{E}_{\mu_\varepsilon}\left[\mathbb{1}_{\mathcal{C}_\Omega} \mid \sigma([0,t]) \right] = C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}) = K_\varepsilon(X^\varepsilon_{[0,t]}).$$

conditional expectation

is a martingale.

<u>Idea</u>. Want in $\varepsilon \to 0$ limit, corresponding conformally invariant martingale.

So need 1) $\varepsilon \to 0$ limit of (#): (weak version).

$$C_\varepsilon(\Omega, a \mid X^\varepsilon_{[0,t]}) = C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}, X^\varepsilon_t)$$

2). Conformal invariance.

<u>Remark</u>. Using RSW estimates, etc., can show:

C_ε "unif. equicont." for ε suff. small

(Joint work w. I. Binder & L. Chayes).

I.e., can take limit of (#).

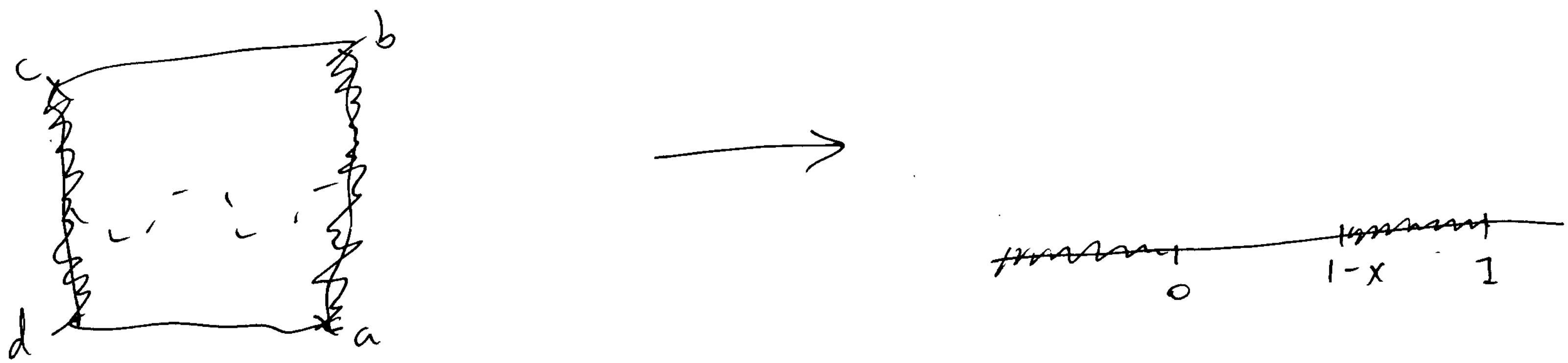
But, no conformal invariance.

(proof complicated...)

Cardy's Formula

$C_\varepsilon \to C_0$, conformally invariant, $\varphi: \Omega_1 \to \Omega_2$.

$$C_0(\Omega_1, \varphi(a), \varphi(b), \varphi(c), \varphi(d)) = C_0(\Omega, a, b, c, d).$$



then

$$C_0(\mathbb{H}, 1-x, 1, \infty, 0) := F(x)$$

$$= \int_0^x [s(1-s)]^{-2/3}\, ds \Big/ \int_0^1 [s(1-s)]^{-2/3}\, ds$$

4 Universal...

Outline of Strategy (Smirnov '06, Werner '07).

(Not all of this new ...).

(0). $\mu_\varepsilon \longrightarrow \mu'$ by Banach-Alaoglu Thm.

(I). Aizenman-Burchard ('99) and a priori estimates:

α). μ' supported on Löewner Curves

β). $\mu_\varepsilon \longrightarrow \mu'$ in sup-norm.

$\left(d(\gamma_1, \gamma_2) = \inf_{\varphi_1, \varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))| \right).$

(II). Take limit of (∗).

(III). Use Cardy's Formula and asympt. expansion at ∞ to conclude $\kappa = 6$.

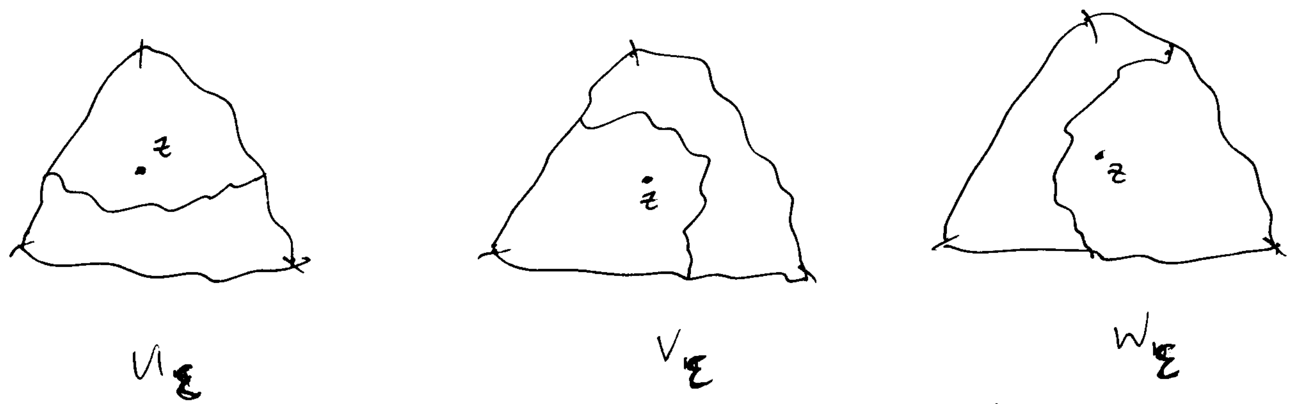
(LERW, '04, Lawler, Schramm, Werner).

Joint work w. I. Binder & L. Chayes:

Carry out strategy for a family of perc. models...

Proof of Cardy's Formula (Smirnov '01).

Consider hexagonal tiling, define

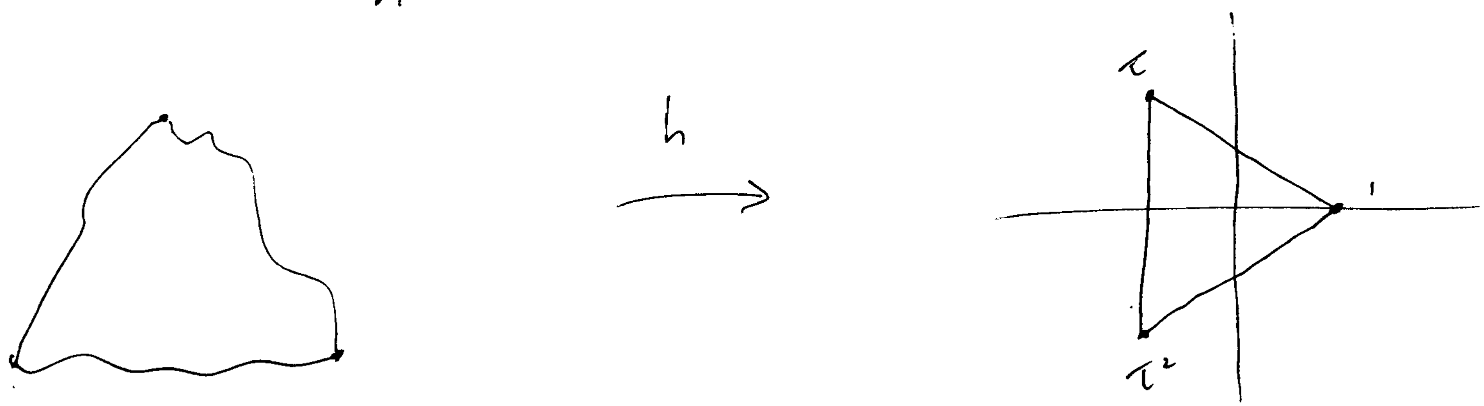


Due to color symmetry, $\tau = e^{2\pi i/3}$

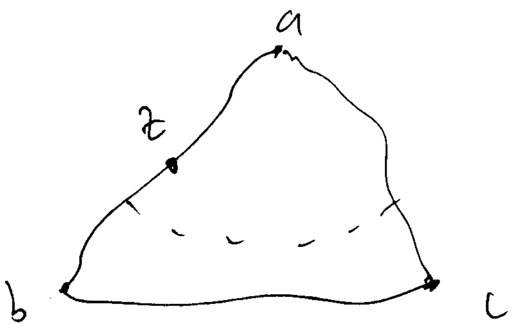
- $\displaystyle\int_{C} u_\varepsilon + \tau v_\varepsilon + \tau^2 w_\varepsilon \, dz \longrightarrow 0$,
- "Obvious boundary values" as $\varepsilon \to 0$.

Specification of limiting function (Beffara '07):

$$h = u + \tau v + \tau^2 w$$



Cardy's Formula = boundary value of u (or v, w).



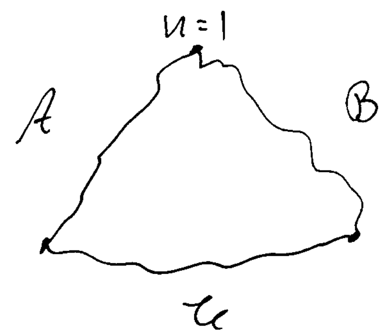
Issues

1). Discretization? Domain? Boundary values?

2). Universality?

Domain Convergence

Have $\quad u_n, v_n, w_n \longrightarrow \underbrace{u, v, w}_{\text{prescribed B.V.}}$



$u = 0$ on $\mathcal{C}$

u non-trivial on $\mathcal{A}, \mathcal{B}$.

(should fix pt. $z_0 \in \Omega_n, \Omega$).

At least want ("pointwise" conditions):

Gives ∃stence of limiting values
- (i_I). $z \in \Omega, \Rightarrow z \in \Omega_n$, n suff. large
- (i_{II}). $z_{n_k} \in \Omega_{n_k}^c$, $z_{n_k} \to z$, $\Rightarrow z \in \Omega^c$

makes sure Ω_n not too large
- (e). $\forall z \in \Omega^c$, $\exists z_{n_k} \in \Omega_{n_k}^c$, $z_{n_k} \to z$.

By Carathéodory's Thm, (i_I), (i_{II}), (e) equiv. to

<u>Carathéodory convergence</u>

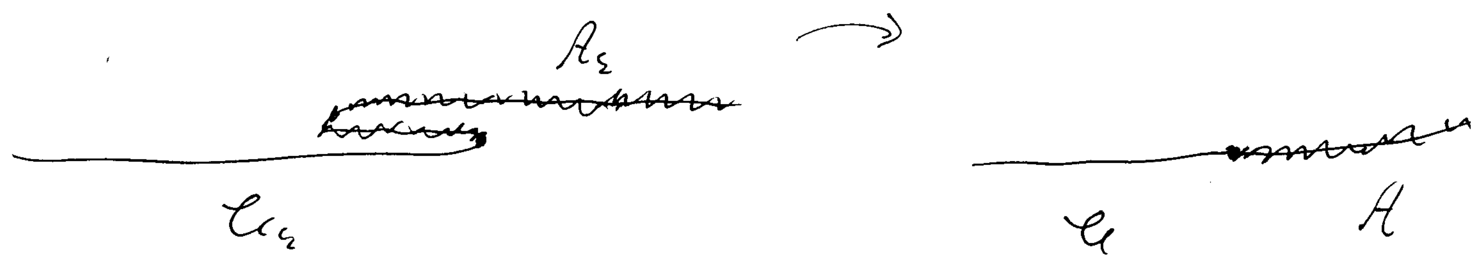
$$\varphi_n \to \varphi \quad \text{locally uniformly}$$

$$\varphi_n : \mathbb{D} \to \Omega_n, \qquad \varphi : \mathbb{D} \to \Omega$$

$$\varphi_n(0) = z_0,\ \varphi_n'(0) > 0; \quad \varphi(0) = z_0,\ \varphi'(0) > 0.$$

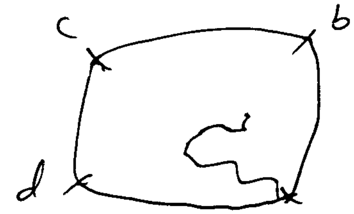
For us not enough, need conv. of separate boundary pieces:

<u>Example</u>. (want $u = 0$ on $\mathcal{C}$, u = nontrivial on $\mathcal{A}$).

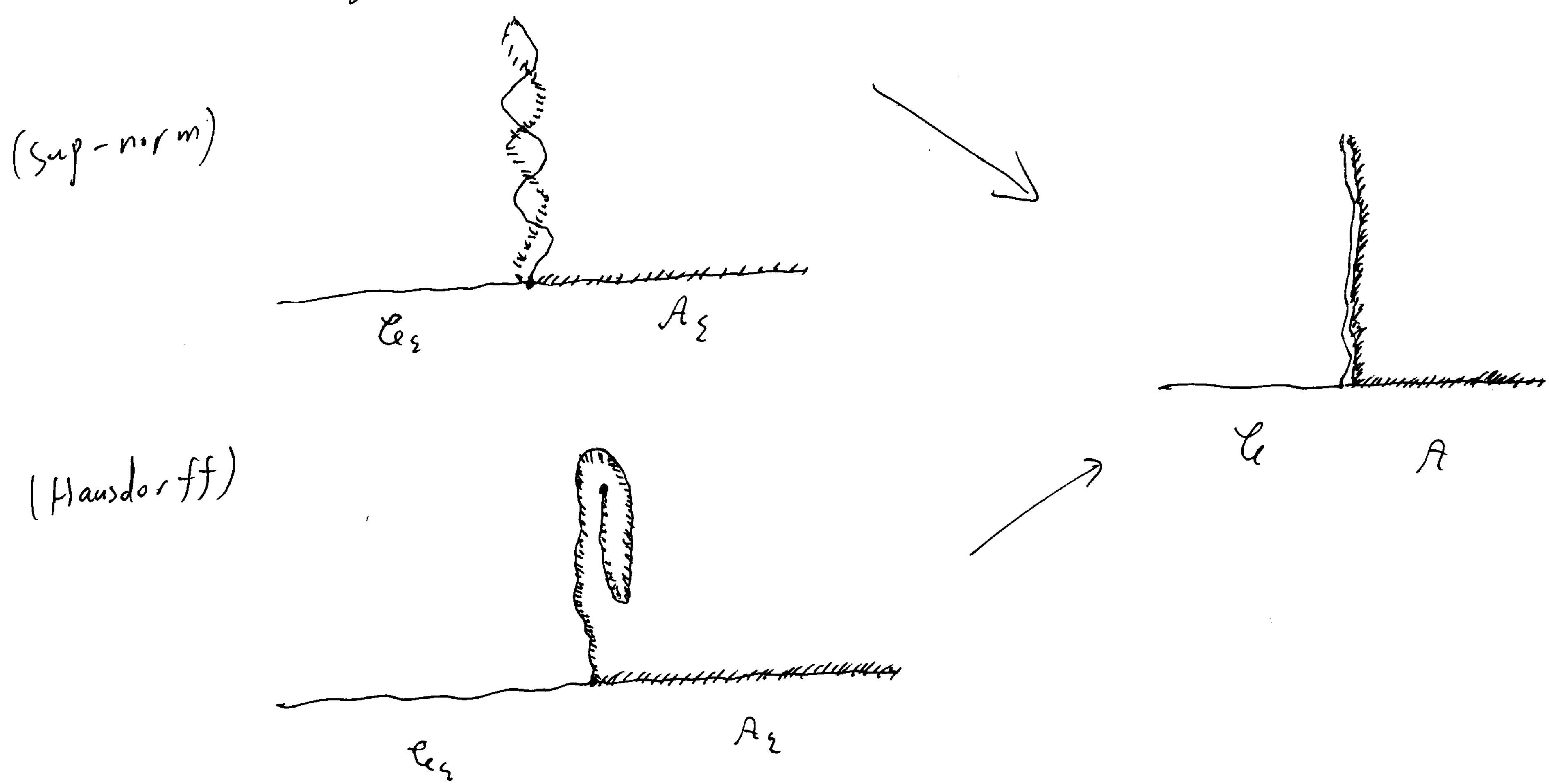


Slit Domains

Need to consider $\Omega \setminus X_{[0,t]}$:



More negative thinking:



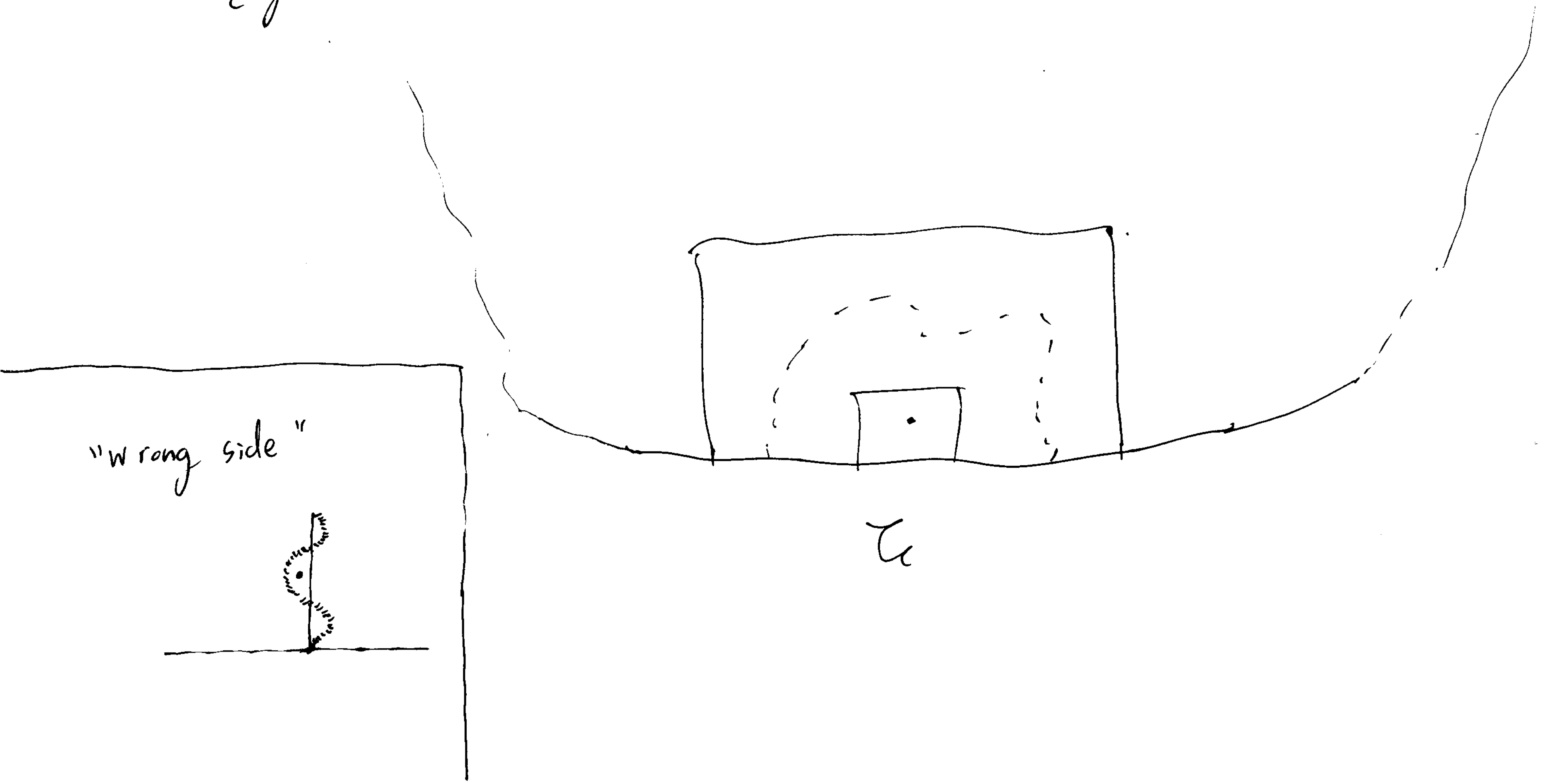
Think positively: Boundary values established by RSW.

so "just" need in the limit:

z "close to" $\mathcal{C}$

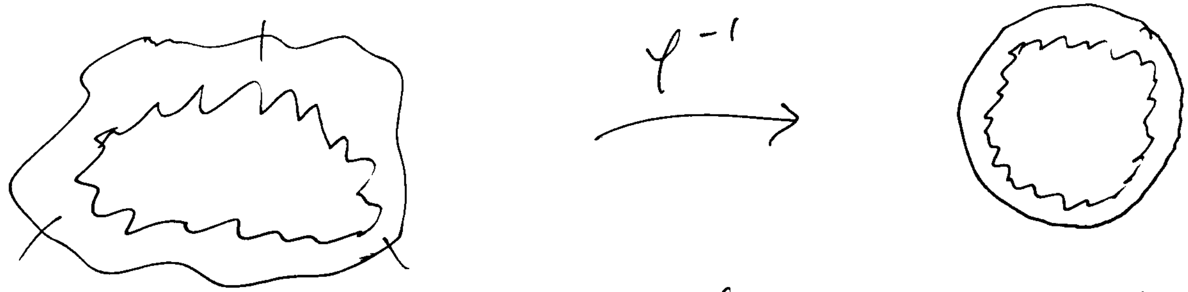
$\Rightarrow z$ "close to" $\mathcal{C}_\varepsilon$, for ε small.

E.g.



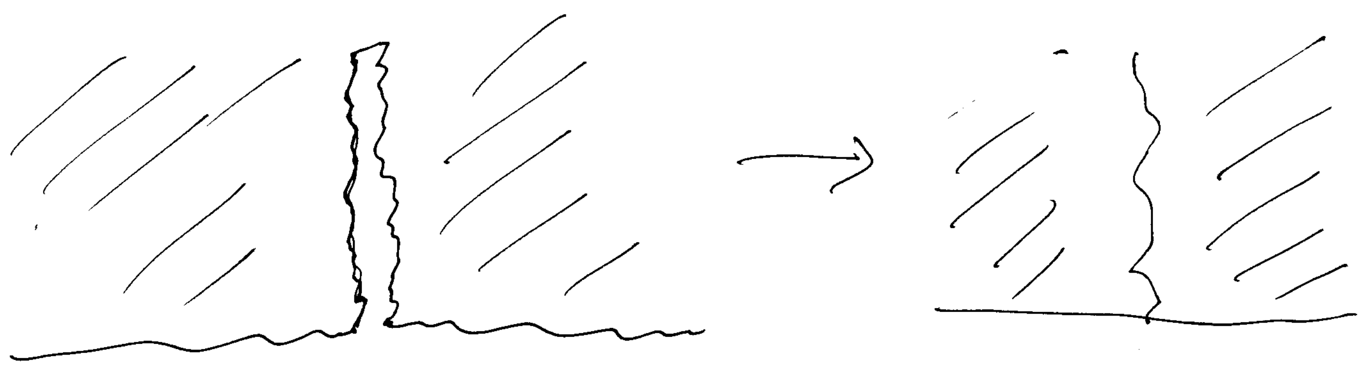
Interior Approximation

Straighten out topology with unif. map. $\Omega_\varepsilon < \Omega$.



If $\Omega_\varepsilon < \Omega$, can view under ONE conformal map.

For slit domains,



But, recall. need

$$C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}, X^\varepsilon_t) \longrightarrow C_0(\Omega \setminus X_{[0,t]}, X_t).$$

where $X^\varepsilon_{[0,t]} \longrightarrow X_{[0,t]}$ in sup-norm.

Need to consider: sup-approximations:

$$\gamma^{(l)}_\varepsilon \to \gamma, \quad \gamma^{(r)}_\varepsilon \longrightarrow \gamma.$$

(could have $\gamma^{(l)}_\varepsilon = \gamma^{(r)}_\varepsilon$)

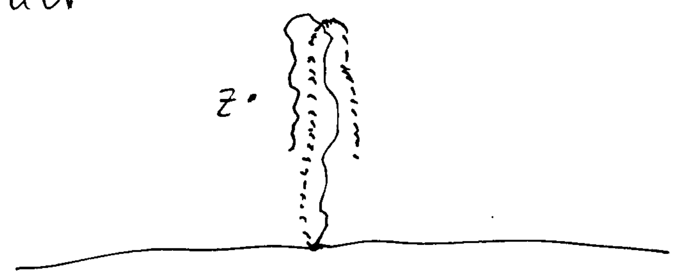
Can show

Lemma. (Topological consistency). If z close to $\mathcal{C}$, then z close to $\mathcal{C}_\varepsilon$ for ε suff. small.

Remark. This is a limiting statement. How small ε has to be depends on

$\mathcal{C}$ and $d(z, \mathcal{C})$.

E.g., consider



— = approx

⋯ = limiting curve.

Here z close to left of ⋯, but right of —.

would need the curves much closer.

(However, this can be quantified

$\rightsquigarrow$ unif. cont. statement for $(\varepsilon \ldots)$.

<u>Conclude</u>. For hexagonal tiling, have

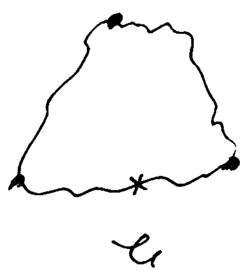
1). Cardy's Formula for general domains.

2). Conv. of interface to SLE_6.

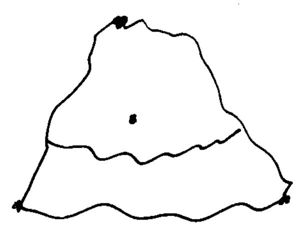
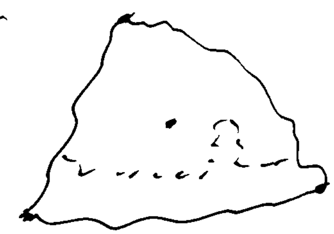
<u>Color Symmetry</u>

Note RSW estimates universal, also e.g.

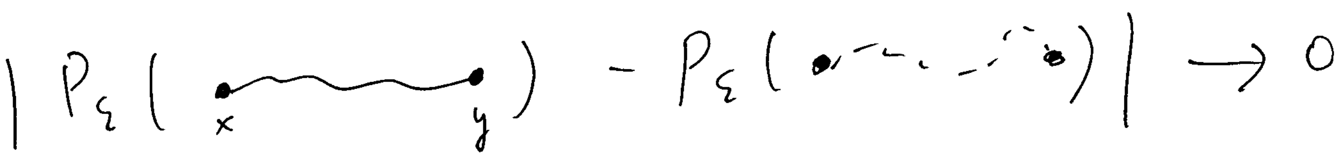
$V^B + W^Y = 1$ on $\mathcal{C}$ by duality.



So boundary value would be universal if had. e.g.

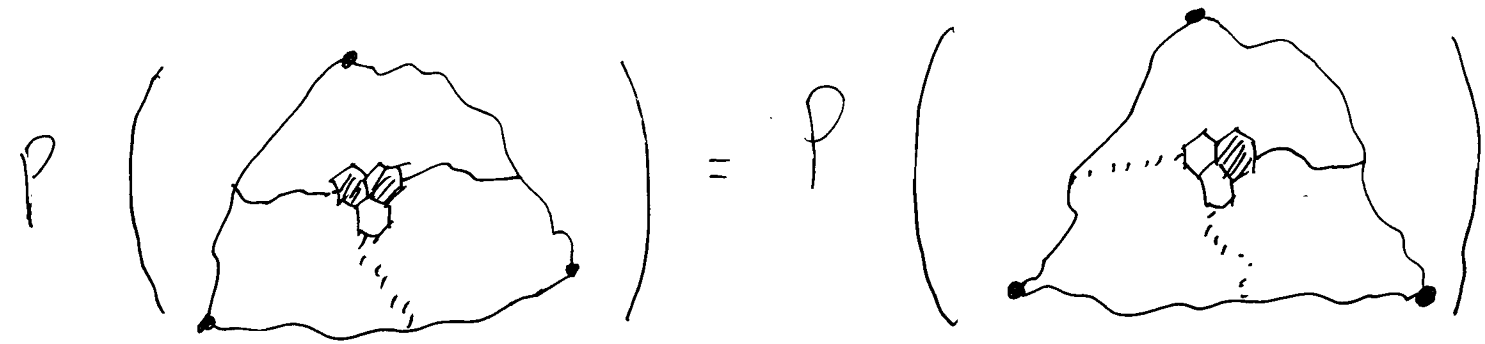
P () = P ().

This would be implied by asymptotic color symmetry, i.e.



as $\varepsilon \to 0$.

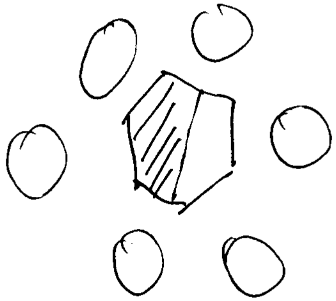
Interior analyticity required more, e.g.,



(some "microscopic" color symmetry).

We have

Joint work with L. Chayes (later also I. Binder)



- parameter s: split some hexagons
- $s = 0$: hexagonal tiling
- locally correlated.

Model has the property, e.g.,

$|$ $P_\varepsilon($  $) - P_\varepsilon($  $)$ $| \to 0$

Had to define e.g., $\tilde{\mu}_\varepsilon$,

1). Expectation of random variable.

2) Enlarged probability space.

Interior analyticity for $\tilde{u}_\varepsilon + \tau \tilde{v}_\varepsilon + \tau^2 \tilde{w}_\varepsilon$, then

$$|\hat{u}_\varepsilon - u_\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The last requires

$$\underbrace{M}_{\text{Minkowski dimension}}(\partial\Omega) < 2$$

<u>Result</u>. For $s \in [0, (2\sqrt{2}+3)^{-1}]$, also have Cardy's Formula and convergence to SLE_6 for Ω with $M(\partial\Omega) < 2$

"universality"

Thank you