Inhomogeneous Continuity Equation with Application to Hamiltonian ODE

(joint work with L. Chayes & W. Gangbo)

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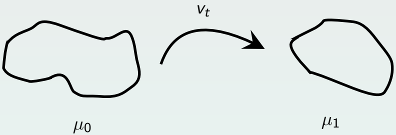
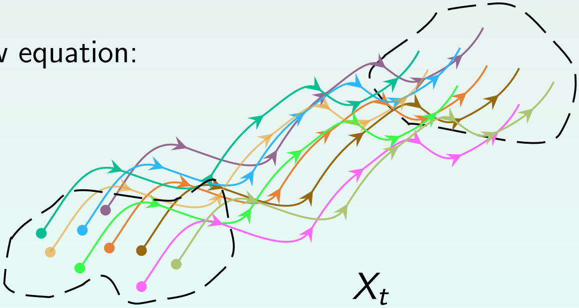
Evolution of Measure

Eulerian:



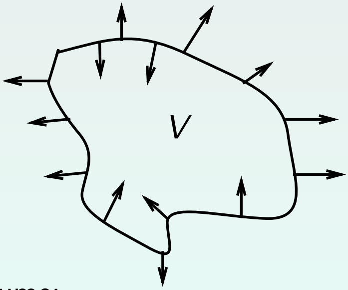
Given v_t, have flow equation:

$$\begin{cases} \dot{X}_t = v_t(X_t) \\ X_0 = \mathrm{id} \end{cases}$$



$\triangle$ in mass = flux in/out of infinitesimal volume:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$



ρ = (probability) density

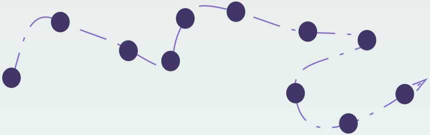
v = velocity field

Integrated version for macroscopic volume:

$$\frac{dM_V}{dt} = \int_V \frac{\partial \rho}{\partial t}\, dx = -\int_V \nabla \cdot (\rho v)\, dx = -\int_{\partial V} \rho\, v \cdot \hat{n}\, dS$$

Continuity Equation II

Mass of particle constant along trajectories (incompressible):



$$\frac{d}{dt}\left[\rho(X_t, t)\right] = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot v = 0.$$

Therefore,

$$\nabla \rho \cdot v = \nabla \cdot (\rho v) \implies \nabla \cdot v = 0$$

and have weak formulation for measures :

$$\partial_t \mu_t + \nabla \cdot (\mu_t v_t) = 0$$

means

$$\int_0^T \int \partial_t \varphi + \langle v_t, \nabla \varphi \rangle \, d\mu_t \, dt = 0 \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0, T))$$

Weak Formulation

Define

$$\mu_t = X_t \# \mu_0$$

(Here $T\#\mu = \nu$ if for any measurable A

$$\nu(A) = \mu(T^{-1}(A))$$

or for any test function $\varphi \in L^1(d\nu)$

$$\int \varphi(y)\, d\nu(y) = \int \varphi(T(x))\, d\mu(x)$$)

Then (formally) $\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0$:

$$\varphi \in C_c^\infty(\mathbb{R}^d \times (0,T)); \quad \Psi(x,t) = \varphi(X_t(x), t)$$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(x) + \langle v_t(x), \nabla \varphi(x) \rangle \, d\mu_t(x)\, dt \\
&= \int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(X_t(x), t) + \langle v_t(X_t(x), \nabla \varphi(X_t(x)) \rangle \, d\mu_0(x)\, dt \\
&= \int_0^T \int_{\mathbb{R}^d} \frac{d\Psi}{dt}(x,t)\, d\mu_0(x)\, dt \\
&= \int_{\mathbb{R}^d} \varphi(X_T(x), T) - \varphi(x, 0)\, d\mu_0(x) \\
&= 0
\end{aligned}$$

Hamiltonian Dynamics I

Let $\mathbb{R}^{2d} \ni x = (p, q) = (\text{momentum, position})$

$$H(p,q) = \frac{1}{2}|p|^2 + \Psi(q) = \textit{kinetic} + \textit{potential}$$

Then
$$\dot{x} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} H_p \\ H_q \end{pmatrix} = \mathbb{J}\nabla H$$

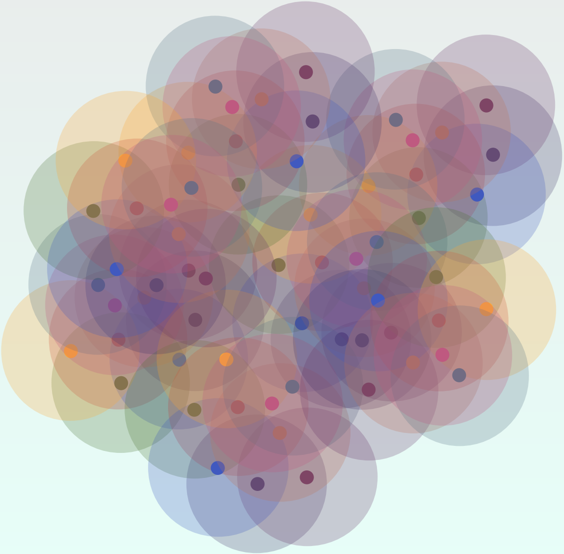
Start with measure, infinite dimensional Hamiltonian system?

$$\mathscr{H}(\mu) = \frac{1}{2}\int |p|^2 \, d\mu + \int \Phi(q) \, d\mu + \frac{1}{2}\int (W * \mu)(q) \, d\mu$$

$$\dot{X}_t = \mathbb{J}[\nabla \mathscr{H}(\mu)](p,q) = (-\nabla(W * \mu + \Phi)(q), p)$$

$\star$ interaction means velocity field has non–trivial dependence on μ_t $\star$

Finite Range Interactions



Hamiltonian Dynamics II

- Infinitesimal conservation of mass certainly holds
- $\nabla \mathscr{H} \perp \mathbb{J}\nabla \mathscr{H} \implies \nabla \cdot (\mathbb{J}\nabla \mathscr{H}) = 0$

Should describe by continuity equation:

$$\partial_t \mu_t + \nabla \cdot (\mathbb{J}\nabla \mathscr{H}(\mu_t)\mu_t) = 0.$$

- Energy not pointwise conserved:

$$\frac{d\mathscr{H}(\mu_t)}{dt}(p,q) = \left[\langle \nabla \mathscr{H}, \mathbb{J}\nabla \mathscr{H}\rangle + \frac{\partial \mathscr{H}}{\partial t}\right](p,q) = \frac{1}{2}\partial_t(W * \mu_t).$$

⋆ Formally, using continuity equation and supposing $|\nabla W| \leqslant B$

$$|\partial_t(W * \mu_t)| = |\frac{d}{dt}\int W(x-y)\ d\mu_t(y)| \leqslant B \int |\mathbb{J}\nabla \mathscr{H}(\mu_t)|\ d\mu_t$$

is *locally* bounded ⋆

Total energy (integrated over μ_t) should still be conserved.

Hamiltonian ODE on Wasserstein Space

L. Ambrosio and W. Gangbo. *Hamiltonian ODE's in the Wasserstein Space of Probability Measures.* Comm. in Pure and Applied Math., **61**, 18–53 (2007).

W. Gangbo, H. K. Kim, and T. Pacini. *Differential forms on Wasserstein space and infinite dimensional Hamiltonian systems.* To appear in Memoirs of AMS.

Definition (Hamiltonian ODE). $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^{2d}) \to (-\infty, \infty]$ (proper, lowersemicontinuous).
A.C. curve $\{\mu_t\}_{[0,T]}$ is a *Hamiltonian ODE* w.r.t. $\mathscr{H}$ if

$$\exists v_t \in L^2(d\mu_t), \quad \|v_t\|_{L^2(d\mu_t)} \in L^1(0,T)$$

such that

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (\mathbb{J} v_t \mu_t) = 0, & t \in (0,T) \\ v_t \in T_{\mu_t}\mathscr{P}_2(\mathbb{R}^{2d}) \cap \partial \mathscr{H}(\mu_t) & \text{for a.e., t} \end{cases}$$

Theorem. (Ambrosio, Gangbo) Suppose $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^{2d}) \to \mathbb{R}$ satisfies

♣$|\nabla \mathscr{H}(x)| \leqslant C(1 + |x|)$

∘ If $\mu_n = \rho_n \mathscr{L}^{2d}, \mu = \rho \mathscr{L}^{2d}$ and $\mu_n \rightharpoonup \mu$ then $\nabla \mathscr{H}(\mu_{n_k})\mu_{n_k} \rightharpoonup \nabla \mathscr{H}(\mu)\mu$

Then given $\mu_0 = \rho_0 \mathscr{L}^{2d}$:

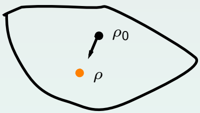
∘The Hamiltonian ODE admits a solution for $t \in [0,T]$

∘ $t \mapsto \mu_t$ is $L(T,\mu_0)$–Lipschitz (with respect to the Wasserstein distance)

∘ If $\mathscr{H}$ is λ–convex, then $\mathscr{H}(\mu_t) = \mathscr{H}(\overline{\mu})$.

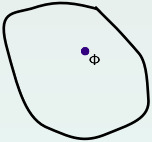
Wasserstein Distance

$\mathcal{M}$
(density)



$\Phi \# \rho_0 = \rho \leftarrow \Phi : \Pi$

$\mathcal{M}*$
(flow map)



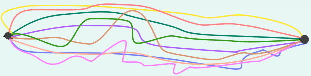
$s = -\nabla \cdot (\rho \nabla p); \quad g_\rho(s_1, s_2) = \int \rho \nabla \rho_1 \cdot \rho_2$

(non-flat)

$g^*_\Phi(v_1, v_2) = \int (v_1 \cdot v_2) \rho_0$

(flat)

(Induced distance:



$d(x_0, x_1)^2 = \inf\{\int_0^1 g_{x(t)}(\frac{dx}{dt}, \frac{dx}{dt})\, dt : t \mapsto x(t) \in \mathcal{M}', x(0) = x_0, x(1) = x_1\}$)

Upshot:

$$d(\rho_0, \rho)^2 = \inf_{\Phi:\rho=\Phi\#\rho_0} \int \rho_0 |\mathrm{id} - \Phi|^2$$

F. Otto.
The geometry of dissipative evolution eqns: the porous medium equation.
Comm. PDE, 26 (2001), 101-174.

A.C. Curves and the Continuity Equation

Definition. Let

$$\mathscr{P}_2(\mathbb{R}^d, W_2)$$

denote the space of probability measures with bounded second moment equipped with the Wasserstein distance

$$W_2^2(\mu,\nu) = \min\left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu) \right\}$$

and

$$\Gamma(\mu,\nu) = \{\gamma : \gamma(A\times\mathbb{R}^d) = \mu(A) \text{ and } \gamma(\mathbb{R}^d\times B) = \nu(B), \text{ for all measurable } A \text{ and } B\}$$

Theorem. There is a correspondence:

$$\{\text{A.C. curves in } \mathscr{P}_2(\mathbb{R}^d, W_2)\} \Longleftrightarrow \{\text{velocity fields } v_t \in L^2(d\mu_t)\}$$

via

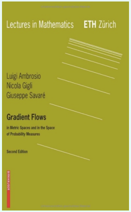
$$\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{1}{|h|}W_2(\mu_{t+h},\mu_t)(\leqslant) = \|v_t\|_{L^2(\mu_t)}$$

Thus

$$W_2^2(\mu_0,\mu_1) = \min\left\{ \int_0^1 \|v_t\|^2_{L^2(d\mu_t)} : \partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0 \right\}$$

and

$$T_\mu\mathscr{P}_2(\mathbb{R}^d, W_2) = \overline{\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(d\mu)}$$



Mass Reaching Infinity in Finite Time

Condition (♣).

We are solving

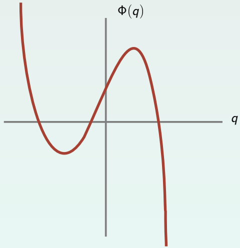
$$\partial_t \mu_t + \nabla \cdot (\mathbb{J} \nabla \mathscr{H} \mu_t) = 0; \quad v_t := \mathbb{J} \nabla \mathscr{H}(\mu_t)$$

Recall characteristics

$$\dot{X}_t = v_t(X_t); \quad X_0 = \mathrm{id}$$

$|v_t(x)| \leqslant C(1 + |x|) \implies |X_t| \lesssim e^{Ct}(1 + |X_0|)$:
preserves compact support, second moment...

What about other Hamiltonians? E.g.,

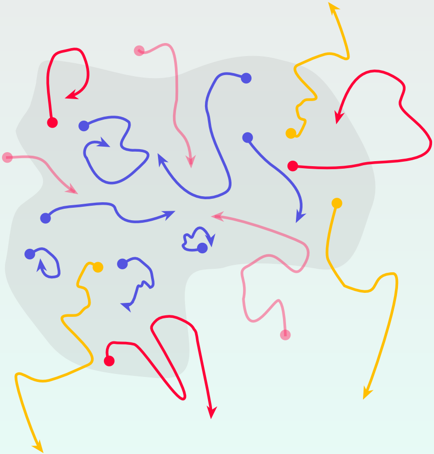


Explicit Computation. $|v_t(X_t)| = C(1 + |X_t|)^R, R > 1$

$$\left(\frac{|X_t|}{|X_0|}\right)^{R-1} = \frac{1}{1 - t(R-1)|X_0|^{R-1}}$$

$$x \rightsquigarrow \infty \quad \text{at time} \quad \tau(x) = \frac{1}{(R-1)|x|^{R-1}} < \infty$$

Continuity Equation in "Finite Volume"



Particles that have ever been in finite region during $[0, t]$:

blue = good
pink = negligible
red = bad
yellow = gone.

Expect. Under reasonable dynamical conditions, still have

$$\partial_t \mu_t + \nabla \cdot (\mathbb{J} \nabla \mathscr{H}(\mu_t) \mu_t) = 0 \qquad \text{distributionally.}$$

Example: Quadratic Velocity in 1D

Consider the velocity field and associated trajectories

$$v_t(x) = x^2, \quad x_t = \frac{x_0}{1 - tx_0}$$

and densities

$$\rho_0 = \mathbf{1}_{[0,1]}, \quad \rho_t = x_t \# \rho_0.$$

By change of variables, have

$$\begin{aligned} \rho_t(y) &= \rho_0(x_t^{-1}(y))(x_t^{-1})'(y) \\ &= \frac{1}{(1+yt)^2}. \end{aligned}$$

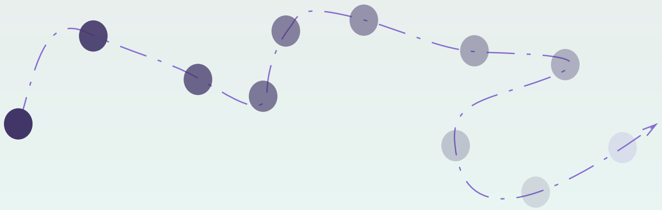
We have then

$$\partial_t \rho_t = \frac{-2y}{(1+yt)^3} \quad \text{and} \quad (\rho_t v_t)' = \frac{2y}{(1+yt)^3}$$

and so

$$\partial_t \rho_t + (\rho_t v_t)' = 0.$$

Regularization: Fade With Arc Length



$$\dot{X}_t = v_t(X_t) \qquad M_t = M_0 e^{-\int_0^t C_s(X_s)|v_s(X_s)| \, ds}$$

For simplicity, $C_s \equiv \varepsilon$; later, send $\varepsilon \to 0$.

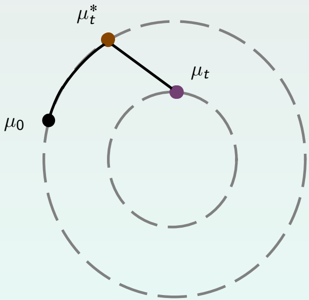
Inhomogeneous Continuity Equation

$$(\spadesuit) \qquad \partial_t \mu_t^\varepsilon + \nabla \cdot (v_t \mu_t^\varepsilon) = -\varepsilon |v_t| \mu_t^\varepsilon$$

Given μ_0, v_t, define

$$(\mu_t^\varepsilon)^* = X_t^\varepsilon \# \mu_0$$

$$R_t^\varepsilon(X_t^\varepsilon) = \exp(-\varepsilon \int_0^t |v_t(X_s^\varepsilon)| \, ds)$$



then

$$\mu_t^\varepsilon = R_t^\varepsilon (\mu_t^\varepsilon)^*$$

satisfies $(\spadesuit)$.

Proposition. $(\spadesuit)$ preserves α–exponential moments for $\alpha \leqslant \varepsilon$, since

distance traveled $\leqslant$ arclength

◇

⋆ directly gives global (in space) regularization ⋆

Existence of ε–Dynamics

Lemma. Let $\mu_0 \in \mathcal{M}_{\infty,\varepsilon}$. Suppose we have prescribed (time–dependent) velocity fields v_t^ε satisfying

$$|v_t^\varepsilon(x)| \leqslant C(1+|x|)^R$$

for some constants $C, R > 0$. Then for $0 < T < \infty$

- $\exists$ distributional solution $(\mu_t^\varepsilon)_{t\in[0,T]}$ to

$$\partial_t \mu_t^\varepsilon + \nabla \cdot (v_t^\varepsilon \mu_t^\varepsilon) = -\varepsilon |v_t^\varepsilon| \mu_t^\varepsilon$$

$\forall \varphi \in C_c^\infty(\mathbb{R}^{2d} \times [0,T])), \quad \int_0^T \int_{\mathbb{R}^{2d}} (\partial_t \varphi + \langle v_t, \nabla_x \varphi \rangle)\, d\mu_t\, dt = -\varepsilon \int_0^T \int_{\mathbb{R}^{2d}} |v_t^\varepsilon| \varphi\, d\mu_t\, dt$

realized as a linear functional such that

$$\int_{\mathbb{R}^{2d}} \varphi\, d\mu_t^\varepsilon = \int_{\mathbb{S}_t^\varepsilon} (R_t^\varepsilon \varphi) \circ X_t^\varepsilon\, d\mu_0, \quad \forall \varphi \in C_c(\mathbb{R}^{2d}).$$

- $(\mu_t^\varepsilon)_{t\in[0,T]}$ is narrowly continuous.
- Preservation of moments.

Topologies of Convergence

$(C_c^\infty \subset) C_c$		C_0		C_b
cpctly supported	$\subseteq$	vanishing at ∞	$\subseteq$	bounded
$\downarrow$		$\downarrow$		$\downarrow$
distributional		weak*		narrow

- finite measures $\implies$ Banach–Alaoglu gives some limit point in weak* topology
- distributional convergence + moment control $\implies$ narrow convergence

We have Radon measures so if $\mu_n \rightharpoonup \mu$ and A is a Borel set

$$\mu(A^\circ) \leqslant \liminf_n \mu_n(A) \leqslant \limsup_n \mu_n(A) \leqslant \mu(\overline{A})$$

Technical Remarks

Continuity. Let $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and suppose $t \to t^\star$.

Then, with $Y_\tau = X_{t^\star} \circ X_\tau^{-1}$,

$$\left| \int \varphi \, d\mu_t^\varepsilon - \int \varphi \, d\mu_{t^\star}^\varepsilon \right| = \left| \int \left[\varphi - \varphi(Y_{t^\star}) \exp(-\varepsilon \int_t^{t^\star} |\dot{Y}_\tau| \, d\tau) \right] d\mu_t^\varepsilon \right|$$

$$\lesssim_{\varphi, v_t^\varepsilon} |t - t^\star| + \varepsilon$$

Limiting Measures. Suppose $\partial_t \mu_t^\varepsilon + \nabla \cdot (v_t^\varepsilon \mu_t^\varepsilon) = -\varepsilon |v_t^\varepsilon| \mu_t^\varepsilon$ for $t \in [0, T]$ and v_t^ε uniformly locally bounded on $[0, T]$.

For $t_k \in \mathbb{Q} \cap [0, T]$, have by Banach–Alaoglu

$$\mu_{t_k}^\varepsilon \rightharpoonup \mu_{t_k}$$

Continuity gives extension to all t. Limiting dynamics later...

Deficient Hamiltonian ODE I

Theorem. Let $\mu_0 \in \mathcal{M}_{\infty,\varepsilon}$ and $0 < T < \infty$. Let

$$\mathscr{H}(\mu) = \frac{1}{2}\int |p|^2 \, d\mu + \int \Phi(q) \, d\mu + \frac{1}{2}\int (W * \mu)(q) \, d\mu$$

such that $|\Phi(q)| \lesssim |q|^R$, some $R > 0$. Then there exists a narrowly continuous path $t \mapsto \mu_t^\varepsilon \in \mathcal{M}_{\infty,\varepsilon}$ such that

$$\partial_t \mu_t^\varepsilon + \nabla \cdot (\mathbb{J}\nabla\mathscr{H}(\mu_t^\varepsilon)\mu_t^\varepsilon) = -\varepsilon|\mathbb{J}\nabla\mathscr{H}(\mu_t^\varepsilon)|\mu_t^\varepsilon.$$

"Proof". Time discretization: $h = 1/n$, $v_k = \mathbb{J}\nabla\mathscr{H}(\mu_{t_k})\mu_{t_k}$

$$\mu_0^{\varepsilon,n} \rightsquigarrow v_0^{\varepsilon,n} \rightsquigarrow \mu_1^{\varepsilon,n} \rightsquigarrow v_1^{\varepsilon,n} \rightsquigarrow \ldots$$

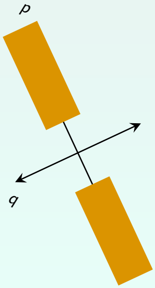
Deficient Hamiltonian ODE II

Get

$$\partial_t \mu_t^{\varepsilon,n} + \nabla \cdot (\mathbb{J}\nabla \mathscr{H}(\mu_{t_n}^{\varepsilon,n})\mu_t^{\varepsilon,n}|) = -\varepsilon|\mathbb{J}\nabla\mathscr{H}(\mu_{t_n}^{\varepsilon,n})|\mu_t^{\varepsilon,n}.$$

Want to take all $n \to \infty$:

- Limiting measure for each t by Banach–Alaoglu
- Only dependence of velocity field on measure is the term $\nabla W * \mu$



- Have tightness by Markov's inequality:

$$\int_{B_r^c \times \mathbb{R}^d} |\nabla W(\bar{q} - q)| \; d\mu_t^{\varepsilon,n}(p,q) \lesssim_W e^{-\varepsilon r} M_\varepsilon(\mu_0)$$

$$\implies \nabla W * \mu_t^{\varepsilon,n} \to \nabla W * \mu_t^{\varepsilon} \quad \text{unif. on cpct sets}$$

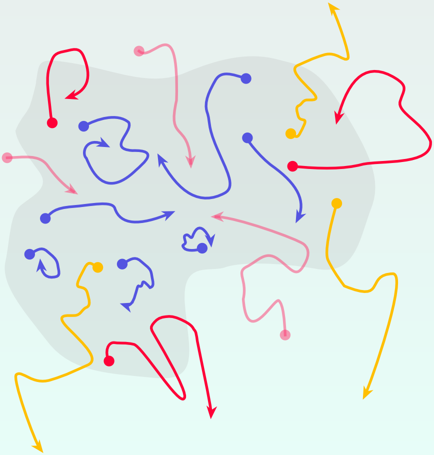
- By deficient continuity equation

$$|\nabla W * \mu_{t_n}^{\varepsilon,n} - \nabla W * \mu_t^{\varepsilon,n}| \lesssim h$$

◇

Uniform in ε Control on Velocity Field

To take $\varepsilon \to 0$ the previous logic can be applied if we can control the velocity field.



Idea: Use the potential Φ

to rid us of red particles.

Enforce that there exists

rings of no return tending

to infinity...

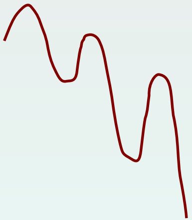
Example: Spherically Symmetric Potential

Consider

$$H(p,q) = \frac{1}{2}|p|^2 + \Upsilon(|q|).$$

Define $\star$–ring by

$$\Upsilon(q) < \Upsilon(L_\star), \quad \text{for all } |q| > |L_\star|.$$



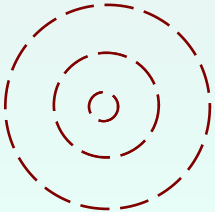
$\star$–rings are rings of no return, since

$$\tilde{H} = \frac{1}{2}\left|\frac{d|q|}{dt}\right|^2 + \Upsilon(q)$$

is *increasing* for $t \geqslant t_\star$:

$$\frac{d\tilde{H}}{dt} = \frac{d|q|}{dt}\left(\frac{d^2|q|}{dt^2} - \frac{d^2 q}{dt^2}\cdot\hat{q}\right) \geqslant 0$$

and at $t_\star$ radial velocity is positive.



More General Potentials

If

$$H(p,q) = \frac{1}{2}|p|^2 + \Phi(q) + \Psi(t,q),$$

with

$$|\nabla\Psi(t,q)| \leqslant B, \quad \text{for all } t, q,$$

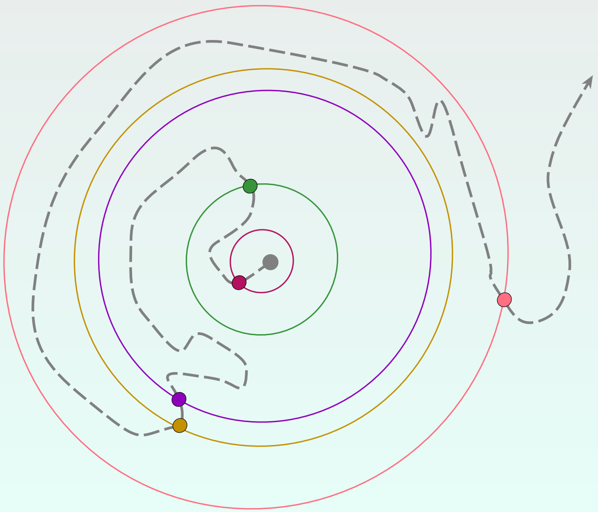
consider bounding potential $u(r)$, such that

$$u'(r) \geqslant B + \max_{|q|=r} \langle \nabla\Phi, \hat{q} \rangle.$$

Then $\star$–rings of u are rings of no return for original dynamics.

Postulate that u has infinitely many $\star$–rings of no return:

Rings of No Return



⋆ c.f., renewal points

for random walks... ⋆

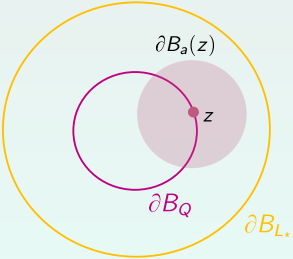
Estimates on Velocity Field I

Choose

$$Q + a < L_\star (\ll r),$$

then $\mathcal{G}$ does not contribute to

$(\nabla W * \mu_t^{\varepsilon,n})(\overline{q})$ for $\overline{q} \in B_Q$:



$\mathcal{S}$ = blue, $\mathcal{G}$ = yellow, $\mathcal{O}$ = pink

- Tightness:

$$\int_{B_r^c \times \mathbb{R}^d} |\nabla W(p, \overline{q} - q)| \, d\mu_t^{\varepsilon,n} \lesssim_W \mu_0((\mathcal{S} \cup \mathcal{O}) \cap \{p_t \geqslant r\})$$

$$\leqslant \mu_0(\mathbb{R}^d \times B_{L_\star}^c) + \mu_0(\mathcal{S} \cap \{p_t \geqslant r\}).$$

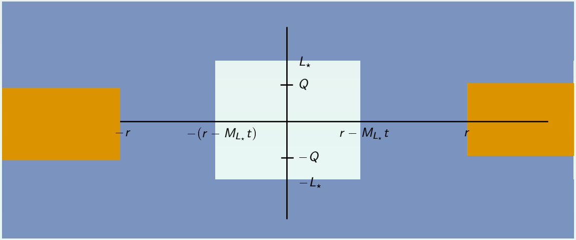
Estimates on Velocity Field II

On $\mathcal{S}$, $|q_s| \leqslant L$, for all $0 \leqslant s \leqslant t$ so

$$r \leqslant |p_t| \leqslant \int \left| \frac{dp_s}{ds} \right| \, ds + |p_0| \leqslant \int |(\nabla \Phi + \nabla W * \mu_t^{\varepsilon,n})(q_s)| \, ds \leqslant M_{L_\star} t + |p_0|,$$

so

$$\mathcal{S} \cap \{p_t \geqslant r\} \subset B^c_{r - M_{L_\star} t} \times B_{L_\star}.$$



- Time evolution: Formally,

$$\partial_t F_t^{\varepsilon,n}(\overline{p}, \overline{q}) = \int_{\mathbb{R}^{2d}} (p \cdot \nabla^2 W(\overline{q} - q) - \varepsilon |v_t^{\varepsilon,n}(p,q)| \nabla W(\overline{q} - q)) \, d\mu_t^{\varepsilon,n}(p,q).$$

This can be estimated a similar way, now invoking moment bound on μ_0...

Hamiltonian ODE I

Theorem. Let $\mu_0 \in \mathcal{M}_{\infty,\alpha}$, some $\alpha > 0$ and $0 < T < \infty$. Let

$$\mathscr{H}(\mu) = \frac{1}{2}\int |p|^2 \, d\mu + \int \Phi(q) \, d\mu + \frac{1}{2}\int (W * \mu)(q) \, d\mu$$

such that $|\Phi(q)| \lesssim |q|^R$, some $R > 0$. Then there exists distributional limit $(\mu_t)_{t\in[0,T]}$ of $(\mu_t^{\varepsilon,n})_{t\in[0,T]}$ along some subsequence (ε_k, n_k) such that

- $t \mapsto \mu_t \in \mathcal{M}$ is distributionally continuous and
- $(\mu_t)_{t\in[0,T]}$ satisfies the continuity equation:

$$\partial_t \mu_t + \nabla \cdot (\mathbb{J}\nabla \mathscr{H}(\mu_t)\mu_t) = 0.$$

◇

Representation Formula

- $\mu_t^{\varepsilon,n}$ defined by pushforward: $\mu_t^{\varepsilon,n} = X_t^{\varepsilon,n} \# \mu_0$, so have representation formula:

$$\int_{\mathbb{R}^{2d}} \varphi(y)\ d\mu_t^{\varepsilon,n}(y) = \int_{\mathbb{S}_t^{\varepsilon,n}} (\varphi \cdot R_t^{\varepsilon,n}) \circ X_t^{\varepsilon,n}\ d\mu_0(x), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^D),$$

where

$$\mathbb{S}_t^\varepsilon = \{x \in \mathbb{R}^D : \exists! \text{ solution to } \dot{X}_s^\varepsilon = v_s^\varepsilon(X_s^\varepsilon), X_0^\varepsilon = x, \forall s \in [0,t]\}.$$

- μ_t^ε, μ_t obtained abstractly, so need to retrieve representation formula...

Need to show

$$\int_{\mathbb{S}_t^{\varepsilon,n}} (\varphi \cdot R_t^{\varepsilon,n}) \circ X_t^{\varepsilon,n}\ d\mu_0 \to \int_{\mathbb{S}_t^\varepsilon} (\varphi \cdot R_t^\varepsilon) \circ X_t^\varepsilon d\mu_0$$

- If $x \in \mathbb{S}_t^\varepsilon$, then $x \in \mathbb{B}_t^\varepsilon(L)$ for L sufficiently large and show pointwise convergence.
- If $x \notin \mathbb{S}_t^\varepsilon$, argue that $(R_t^{\varepsilon,n} \circ X_t^{\varepsilon,n})(x) \to 0$ as $n \to \infty$.

Both cases follow from finite volume convergence of trajectories:

Finite Volume Closeness of Trajectories I

Lemma. Let $T > 0$. Suppose $v^n \to v$ uniformly on $K \times [0, T]$ for any compact $K \subset \mathbb{R}^D$ and

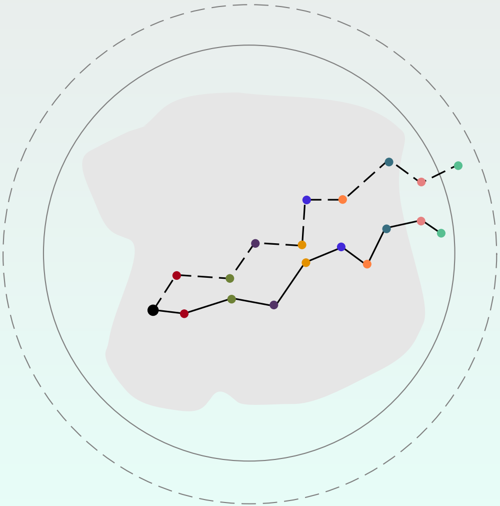
$$\sup_n \left[\sup_{t \in (0,T)} \sup_{x \in K} |v_t^n(x)| + \mathrm{Lip}(v_t^n, K) \right] := f_K < \infty.$$

Then given any $\delta > 0$

$$\sup_{x \in \mathbb{B}_t(L)} \sup_{s \in [0,t]} |X_s^n - X_s(x)| < \delta$$

for n sufficiently large, where

$$\mathbb{B}_L(t) := \{x : X_s(x) \in B_L, \ \forall s \in [0,t]\}$$
$$(\subset \mathrm{supp}(\mu_0)).$$



Finite Volume Closeness of Trajectories II

- For n sufficiently large so that $|v^n - v| < \sigma$ and $X_s \in B_L, X_s^n \in B_{L+\delta}$,

$$\begin{aligned}\frac{d}{ds}|X_s^n - X_s| &\leqslant |v_s^n(X_s^n) - v_s(X_s)| \\ &\leqslant |v_s^n(X_s^n) - v_s^n(X_s)| + |v_s^n(X_s) - v_s(X_s)| \\ &\leqslant \|v_s^n\|_{\text{Lip}} \cdot |X_s^n - X_s| + \sigma \\ &\leqslant f_{B_{L+\delta}}|X_s^n - X_s| + \sigma.\end{aligned}$$

- By Gronwall and choosing σ sufficiently small (n sufficiently large)

$$|X_T^n - X_T| \leqslant \frac{\sigma}{f_{B_{L+\delta}}} \cdot e^{f_{B_{L+\delta}} T} < \delta.$$

- Result follows by a bootstrapping argument.

$\diamond$

Representation formula holds for μ_t^ε and can directly take $\varepsilon \to 0$:

Hamiltonian ODE II

Theorem. Let $\mu_0 \in \mathcal{M}_{\infty,\alpha}$, some $\alpha > 0$ and $0 < T < \infty$. Let

$$\mathscr{H}(\mu) = \frac{1}{2}\int |p|^2 \, d\mu + \int \Phi(q) \, d\mu + \frac{1}{2}\int (W * \mu)(q) \, d\mu$$

such that $|\Phi(q)| \lesssim |q|^R$, some $R > 0$. Then there exists distributional limit $(\mu_t)_{t\in[0,T]}$ of $(\mu_t^{\varepsilon})_{t\in[0,T]}$ along some subsequence (ε_k) such that

- $t \mapsto \mu_t \in \mathcal{M}$ is distributionally continuous and
- $(\mu_t)_{t\in[0,T]}$ satisfies the continuity equation:

$$\partial_t \mu_t + \nabla \cdot (\mathbb{J}\nabla \mathscr{H}(\mu_t)\mu_t) = 0.$$

◇

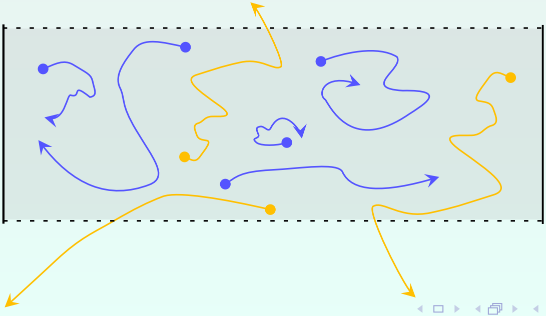
Phase Space Regions of No Return

Let $L_\star$ correspond to $\star$–ring and define

$$\bar{\Omega}_{L_\star}(t) = B_{L_\star+(a_\star+\eta)t} \times B_{L_\star},$$

where $\eta > 0$, $a_\star = \sup_{q \in B_{L_\star}} |\nabla\Phi| + |\nabla W|$, so that $\frac{d}{ds}|p_s| \leqslant a_\star, \forall s \in [0, t]$.

Then:



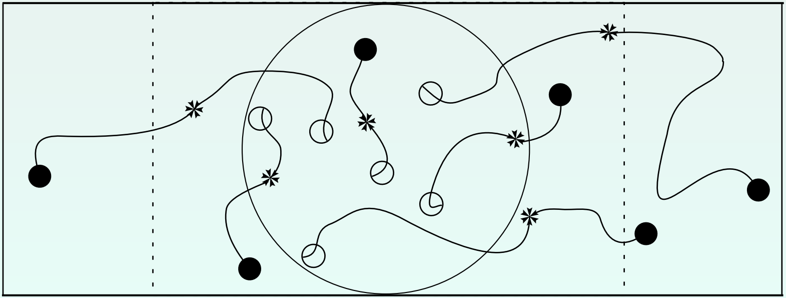
Monotonicity of Mass

Let $0 \leqslant t_1 < t_2$. Given any $\delta > 0$, let $L_\star$ be such that

$$\mu_0(B_{L_\star}) < \delta.$$

Then can show for all $\varepsilon \geqslant 0$,

$$\mu^{\varepsilon}_{t_1}(\bar{\Omega}_{L_\star}(t_1)) \geqslant \mu^{\varepsilon}_{t_2}(\bar{\Omega}_{L_\star}(t_2)) - \delta.$$



⋆ Could also directly obtain representation formula for μ_t by invoking no return property... ⋆

Mass Convergence?

"mass difference = mass "burned" at ∞ by ε regularization"

$$M_0 - M_t = M_0 - \lim_{\varepsilon \to 0} M_t^\varepsilon$$

∘ Since the function $1 \equiv f \notin C_c$, mass convergence not immediate.

• Without interaction W, trajectories same for all $\varepsilon \implies$ mass convergence: Have $M_t^\varepsilon \nearrow M_t^*$ is well defined. Let $\delta > 0$.

(i) Choose L such that $\mu_t(B_L^c) < \delta$. Then

$$M_t \leqslant \mu_t(B_L^\circ) \leqslant \liminf \mu_t^\varepsilon(B_L) + \delta \leqslant M_t^* + \delta.$$

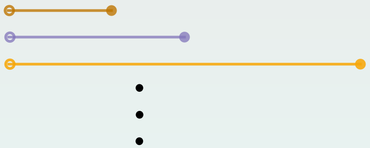
(ii) For any $\varepsilon > 0$, choose L_ε such that $\mu_t^\varepsilon(B_{L_\varepsilon}) < \delta$. Then

$$M_t \geqslant \mu_t(B_{L_\varepsilon}) \geqslant \mu_t^\varepsilon(B_{L_\varepsilon}) \geqslant M_t^\varepsilon - \delta.$$

Presence of $W \implies$ non–trivial dependence of trajectories on *measure* so *a priori*:

“Counterexample” to Mass Convergence I

Varying ε:

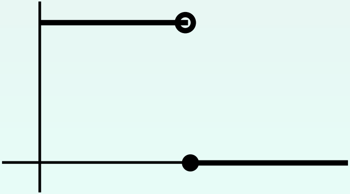


distance $= \lambda\ \varepsilon^{-1}$

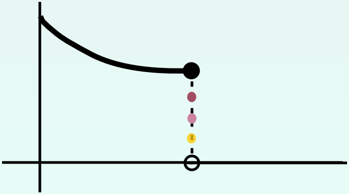
mass $= 1$ at time 0

mass $= e^{-\lambda}$ at time 1

M_t:



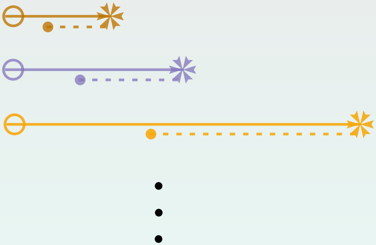
M_t^ε:



⋆ Mass does not converge at point of discontinuity... ⋆

"Counterexample" to Mass Convergence II

Varying ε:

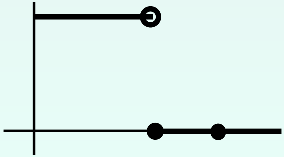


mass $= 1$ at time 0

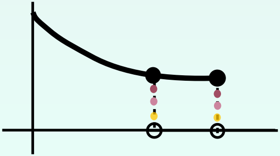
mass $= e^{-\lambda}$ at time $t - \tau$

mass > 0 at time t

M_t:

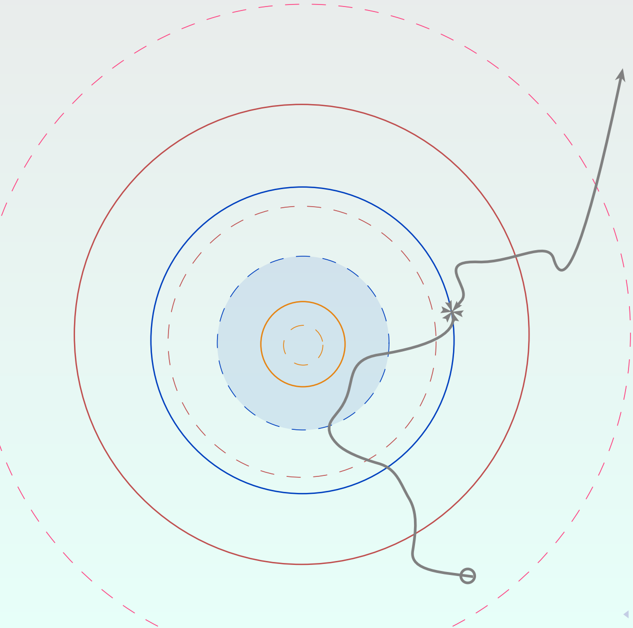


M_t^{ε}:



Mass not tending to ∞ fast enough:

Stronger Dynamical Condition



- $\ell(L) < L$ ring of no return, $\ell, L \to \infty$;
  - $$E_L(t) = \{q_0 : q_t \in \partial B_L, q_{t'} \in B_{\ell(L)} \text{ for some } t' < t\};$$
  - $$\theta_L(t) = \sup\{\tau : q_0 \in E_L(t), |q_{t+\tau}| < \infty\};$$
  - $$\tau_L = \sup_t \theta_L(t);$$
- Require:
  $$\lim_{L\to\infty} \tau_L = 0.$$

Example: Super–quadratic Potential

Consider $\Upsilon(q) \sim -|q|^{1+R}$, $R > 1$. Recall

$$\tilde{H} = \frac{1}{2}\left|\frac{d|q|}{dt}\right|^2 + \Upsilon(q)$$

is increasing provided $\frac{d|q|}{dt} > 0$. Therefore (for $|q| \gg 1$)

$$\frac{d|q|}{dt} > \sqrt{2(\tilde{H}_0 - \Upsilon(q))} \sim |q|^{\frac{1+R}{2}} := C(1+|q|)^s, \quad s > 1.$$

Suppose at time t, $|q_t| = L_\star$, $\frac{d|q_t|}{dt} > 0$. Direct integration of differential inequality:

$$(1+|q_{t+\tau}|)^{s-1} \geqslant \frac{(1+|q_t|)^{s-1}}{1 - C\tau(s-1)(1+|q_t|)^{s-1}}.$$

We conclude the particle reaches infinity by time $t + \tau_{L_\star}$, where

$$\tau_{L_\star} \sim \frac{1}{L_\star^{s-1}} \to 0 \quad \text{as} \quad L_\star \to \infty$$

Mass Convergence Almost Everywhere

Theorem. Suppose the stronger dynamical condition holds and suppose $\mu_t^\varepsilon \rightharpoonup \mu_t$. Let

$$M_t^- = \lim_{t' \nearrow t} M_{t'}, \quad M_t^+ = \lim_{t' \searrow t} M_{t'}$$

$$M_t^\bullet = \overline{\lim}\, M_t^\varepsilon, \qquad M_t^\circ = \underline{\lim}\, M_t^\varepsilon.$$

Then

$$M_t^+ \leqslant M_t^\circ \leqslant M_t^\bullet \leqslant M_t^-.$$

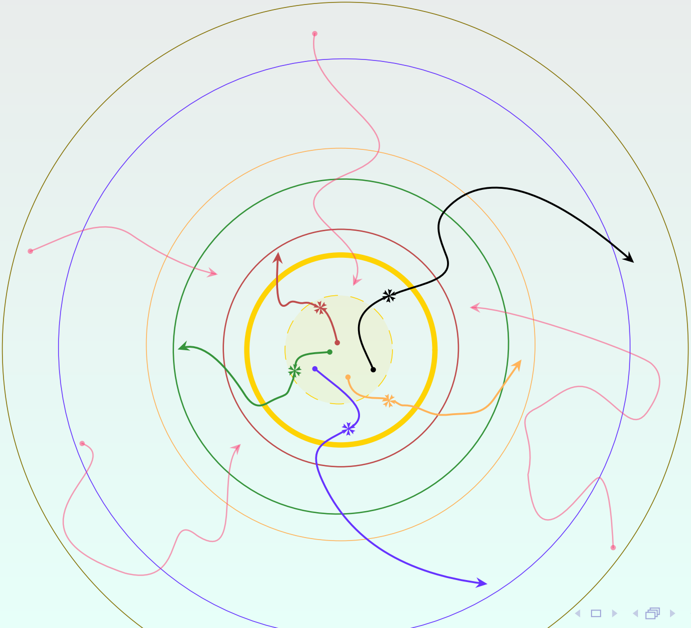
In particular, the mass converges at all points of continuity of M_t.

"Proof." Already have $M_t^+ \leqslant M_t^\circ$. To show $M_t^\bullet \leqslant M_t^-$:

- Let $\delta > 0$ and $\ell = \ell(L)$ be such that $\mu_0(\overline{\bar{\Omega}_\ell(0)^c}) < \delta$.
- For any $\varepsilon > 0$ let $0 < L_\varepsilon < \infty$ be such that $\mu_t^\varepsilon(\overline{\bar{\Omega}_{L_\varepsilon}(t)^c}) < \delta$.

$$M_t^\varepsilon \leqslant \mu_t^\varepsilon(\bar{\Omega}_{L_\varepsilon}(t)) + \delta \leqslant \mu_{t-\tau_L}^\varepsilon(\bar{\Omega}_L(t-\tau_L)) + 2\delta:$$

"One Ring to Rule Them All"



$\varepsilon \longrightarrow 0$

$L \longrightarrow \infty$

Questions and Extensions.

- Meaningful physical systems of relevance?
- Different inhomogeneous equation?
- Uniqueness of limiting measures? (under investigation)
- Stronger topology?

Thank you