A Universality Result for 2D Percolation Models

L. Chayes[1] and H. K. Lei[1]

[1] *Department of Mathematics, UCLA, Los Angeles, CA*

This abstract summarizes the talk concerning [1] that was delivered by one of us (L.C.) at the 2006 Oberwolfach meeting *Spacial Random Processes and Statistical Mechanics.* The starting point of the research – from a certain perspective – is the seminal result of S. Smirnov [4]. In this work, Smirnov showed that the so–called *Carleson–Cardy functions*, functions related to crossing probabilities in a (conformal) triangle, had a particular scaling limit: They are harmonic functions obeying certain boundary conditions on any triangle. These boundary conditions are asymptotically satisfied in the percolation problem defined on the discretization of any triangle and hence, if some version of harmoniticity or analyticity can be established at the discrete level, conformal invariance follows. Smirnov addressed this problem for the critical site percolation model on the triangular lattice. For this geometry, an approximate version of the Cauchy–Riemann equations among the triple of the Cardy–Carleson functions was demonstrated by exploiting the exact color symmetry of the random hexagon tiling realization of this problem.

But the purported "power" of scaling theory for critical models is the notion that *all* models of a similar type should have the same scaling behavior – universality. Till now, there has been little substantive progress in this direction; the talk concerned a (modest) example where this form of universality was demonstrated to be the case.

From our perspective, this work began a while ago with the investigation of bond percolation – and general q–state Potts and random cluster models – on the triangular *bond* lattice [2]. The relevant ingredient from [2] is a straightforward perspective on the somewhat mysterious duality relation for the bond problem on this lattice. The key is to abandon independence on any single triangle while keeping disjoint up–pointing triangles independent. Thus, as far as connectivity is concerned, there are only five (rather than eight) relevant configurations a single such triangle: full, empty, and three single bond events. Parametrizing the model in terms of the respective probabilities of these events a, e, and s, with $a+e+3s = 1$, it is not hard to see that the duality/criticality condition is simply $a = e$. (Much of this had been realized in the physics community, especially in the context of spin systems where the local correlations are represented by three–body interactions. See [5] and the references therein.) The desirable feature of this perspective is that the breakdown maps directly into a modified hexagon tiling of the plane. The "a" and "e" configurations correspond to *yellow* and *blue* hexagons while the three "s" configurations correspond to three half yellow and half blue configurations in which the hexagon is split at the midpoints of two opposing edges. It is noted that *not* all of the possible splits appear, ergo the model does *not* enjoy full color reversal symmetry. This and other features render the full bond problem too hard, at least for the present, and we have to make certain modifications.

The model we treat will be described presently; first let us first fix some parlance. We call a hexagon surrounded by 6 other hexagons a *flower*. We call the middle hexagon an *iris* and the six surrounding hexagons *petals*. Here we first restrict attention to configurations where the hexagons allowed to exhibit the mixed configuration are insulated from each other and second we introduce local correlations forbidding certain configurations on the flowers. Explicitly, our model is defined as follows: Tile the plane with hexagons, some of which are designated to be irises in such a way that flowers are disjoint. The iris can be yellow, mixed or blue with the appropriate probability, petals and all other hexagons are only allowed to be pure blue or pure yellow. Finally, in certain *triggering* configurations, when there are exactly three blue petals, exactly two of which are contiguous, we forbid the iris from exhibiting the mixed configuration.

What has been accomplished with the flower model is the restoration of a strictly local version of self–duality: E.g., if one asks that a designated subset of petals be blue and blue connected inside the flower, then that has the same probability as if one asked the question in yellow. Furthermore, the model enjoys many of the advantages afforded by a hexagon tiling of the plane. The work therefore divided into two episodes which in practice were not disjoint: First establishing that the model as described display the standard features of a lattice percolation model at criticality and second using these critical ingredients and the underlying hexagonal/triangular geometry in an adaptation of the methods of [4].

The crucial device, for both of the episodes, is an object that is termed a *path designate*, which, in a statistical sense, replaces the notion of a microscopic path. A path designate is, as a geometric object, a collection of paths. These paths all agree with each other (and the microscopic definition) on the complement of flowers. However, within the flowers, only the entrance and exit petals are specified along with the requirement that they be somehow connected within. Therefore, two paths in the same path designate which goes through the same flower may exhibit *different* microscopic connections within the flower. The path designation event, which comes in two colors, is that some microscopic realization of the path designate is monochrome. Since the existence of a path designation event is equivalent to the existence of an actual microscopic connection, a preliminary result which goes a long way towards ingredients necessary for both episodes is the following:

Theorem 1. Consider the model as described above with arbitrary placement of the flowers. Let $\mathbf{r}$, $\mathbf{r}'$ denote points (hexagons) which are not irises. Then the probability of a connection between $\mathbf{r}$ and $\mathbf{r}'$ is the same in blue as it is in yellow.

Theorem 1 goes much of the distance to proving the critical ingredients, but two additional difficulties had to be overcome: The RSW lemmas and the FKG property. With regards to the former, which demonstrates a weak scale invariance of crossing probabilities, we are forced into periodic arrangements of flowers so as to use the standard arguments [3] to establish these properties. An input to the RSW lemmas is a set of correlation inequalities known as the FKG property. For independent percolation models, these inequalities hold for all increasing events, a property that was irrevocably lost when we introduced the triggers. However, under the assumption $ae \geq 2s^2$ (a condition that was borrowed from [2]), we are able to prove a curtailed but sufficient version of the FKG property, which holds for path type events.

As for episode two, the fundamental difficulty in extending the result of [4] to other lattices is to establish the so–called Cauchy–Riemann relations which relate pieces of the discrete derivatives of the Cardy–Carleson functions to one another. These relations were established in [4] via the ability, at the microscopic level, to switch colors without changing the probabilities of events. In particular, the needed extension of Theorem 1 is to establish that probabilities of connections are the same for both colors in the (conditioned) presence of paths of fixed color. It is easy to demonstrate that this is patently false for the model at hand. New machinery has to be developed.

Conditional color symmetry in our model is disrupted on the level of single flowers: In particular this means that the probability of, say, a connection from one petal to another in the presence of conditioned petals has a bigger probability in blue than it does in yellow. (The conditioned petals are fixed and may be blue or yellow; ostensibly they belong to another path.) If the conditioned petals are blue, then we may forbid, as a random event, any blue path from touching them. The probability of this event is set so as to lower the overall probability of a blue "transmission" down to that of a yellow. On the other hand, if the conditioned petals are yellow, we could allow a yellow path to share the conditioned petals, again with the appropriate probability. Ultimately what is needed is a set of random variables, whose distribution is strongly coupled with the underlying percolation configurations, which provide or deny "permissions" for transmissions in these conditioned situations. We call these $*$–rules. The upshot is there is a theorem along the lines of Theorem 1, but now the "connection" takes place in the presence of pre–conditioned paths. As is clear from the above discussion, the definition of connection, disjoint, etc. has to be rethought with the notion of these permissions taken into account.

The penultimate step is to substitute the $*$–ruled versions of path events in the definition of the Cardy–Carleson functions. Some work is required to show that these $*$–functions obey the appropriate Cauchy–Riemann relations (boundary conditions are still satisfied) but they are not, after all, exactly the objects of interest. However, a detailed analysis of the configurations which lie in the symmetric difference of the starred and unstarrred versions show that the difference between the two vanishes (in the L^1 sense) as the mesh scale tends to zero. The desired result is therefore recovered for the original Cardy–Carleson functions.

References

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