Convergence to SLE_6 for Percolation Models (joint with I. Binder & L. Chayes)

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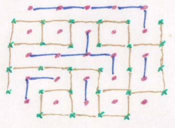
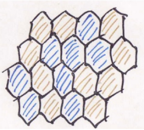
Setup and Scaling Limit

1. $\Omega \subset \mathbf{R}^2$ with $M(\partial\Omega) < 2$
   $\left(M(\partial\Omega) = \limsup_{\vartheta \to 0} \frac{\log \mathcal{N}(\vartheta)}{\log(1/\vartheta)}\right)$

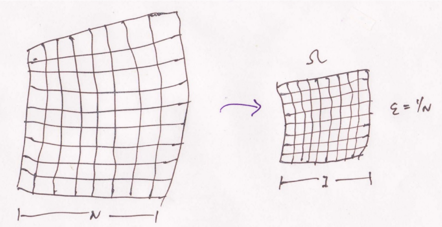
2. Tile with some regular lattice at scale ε





3. Perform percolation at criticality

4. Taking scaling limit: $\varepsilon \to 0$

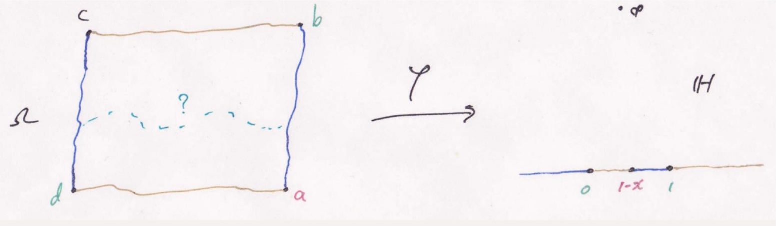


5. Crossing probability? Law of interface?

Conformal Invariance & Cardy's Formula

Conformally invariant: $\varphi : \Omega_1 \to \Omega_2$

$$C_0(\Omega_2, \varphi(a), \varphi(b), \varphi(c), \varphi(d)) = C_0(\Omega_1, a, b, c, d)$$



$$F(x) := C_0(\mathbf{H}, 1-x, 1, \infty, 0) = \frac{\int_0^x [s(1-s)]^{-2/3}\, ds}{\int_0^1 [s(1-s)]^{-2/3}\, ds}$$

Should be lattice independent, but so far:

- Smirnov (2001)
- Camia, Newman, Sidoravicius (2001, 2003)
- Chayes & Lei (2007)

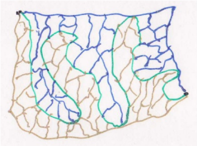
Statement of Result

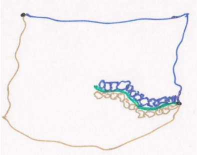
Theorem

Let Ω and Ω_ε be as described. Let $a, c \in \partial\Omega$ and set boundary conditions on Ω_ε so that the Exploration Process runs from a to c. Let μ_ε be the probability measure on curves inherited from the Exploration Process, and let us endow the space of curves with the (weighted) sup–norm metric. Then, under reasonable assumptions on the percolation model,

$$\mu_\varepsilon \Longrightarrow \mu_0,$$

where μ_0 has the law of chordal SLE_6.



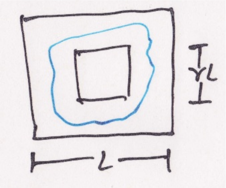


If γ_1, γ_2 are two curves, then the sup–norm is given as

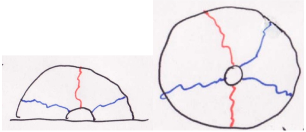
$$\text{dist}(\gamma_1, \gamma_2) = \inf_{\varphi_1, \varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|$$

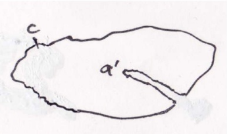
Percolation Assumptions

- RSW theory & FKG inequalities: Scale–invariant bounds on existence of ring in annuli
- BK–type inequalities: $\mathbf{P}(A \circ B) \leq \mathbf{P}(A)\mathbf{P}(B)$
- Universal multi–arm estimates:
  - full–space 5–arm
  - half–space 3–arm
- Definition of Exploration Process leading to a class of admissible domains: the class is closed under deletion of initial portion of explorer path ($M(\partial\Omega) < 2$ is preserved)
- Cardy's Formula for admissible domains



$0 < C_1(\gamma) \leq \mathbf{P}_\gamma(L) \leq C_2(\gamma) < 1$





Schramm's Principle

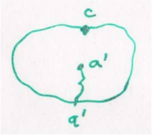
(I) Conformal Invariance

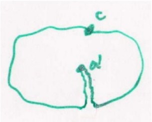
$$\varphi : \Omega \to \varphi(\Omega)$$

then

$$\varphi\#\mu(\Omega, a, c) = \mu(\varphi(\Omega), \varphi(a), \varphi(c))$$

(II) Domain Markov Property





$$\mu(\Omega, a, c)\,|_{\gamma'} = \mu(\Omega \setminus \gamma', a', c)$$

*** law for random curves satisfies (I) & (II) $\Longleftrightarrow$ SLE$_\kappa$ ***

Framework: LSW, ’04 & Smirnov, ’06

1. Show any limit point is supported on Löewner curves
   - view μ_ε as measures on compact $\subset \overline{\Omega}$ gives some limit point
   - Aizenman–Burchard (1999) gives limit supported on curves (BK is useful here)
   - 5–arm and 3–arm estimates used to show limit supported on Löewner curves

   now can describe limit via Löewner evolution with random $w(t)$
2. Take limit of Crossing Domain Markov Property

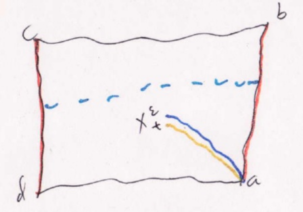
$$C_\varepsilon(\Omega \setminus X_{[0,s]}, X_s, b, c, d) = \mathbf{E}_{X_{[s,t]}}[C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}, X^\varepsilon_t, b, c, d) \mid X_{[0,s]}]$$

3. Expand at ∞ to learn $\kappa = 6$
   - $|C_0(\Omega_s, X_s, b, c, d) - E_{\mu'}[C_0(\Omega_t, X_t, b, c, d) \mid X_{[0,s]}]| \leq \text{error}$
   - conformal map to $\mathbf{H}$:
     $\left| F\left(\frac{g_s(b)-w(s)}{g_s(b)-g_s(d)}\right) - E_{\mu'}\left[F\left(\frac{g_t(b)-w(t)}{g_t(b)-g_t(d)}\right) \mid X_{[0,s]}\right]\right| \leq \text{error}$
     no Domain Markov Property yet
   - Expand g_t at ∞, Taylor expand F :
     $E(w(t) \mid w(s)) = w(s), E(w(t)^2 - 6t \mid w(s)) = w(s)^2 - 6s$
     Lévy’s characterization implies $\kappa = 6$

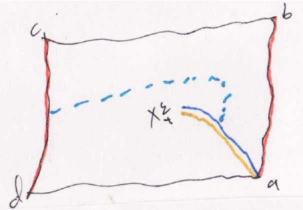
   uses conformal invariance and exact form of Cardy’s Formula

Crossing Domain Markov Property

Ask for conditional crossing probability, then either



or



In either case, have crossing in the corresponding slit domain, so

$$C_\varepsilon(\Omega, a \mid X_{[0,t]}) = C_\varepsilon(\Omega \setminus X_{[0,t]}, X_t^\varepsilon)$$

Using two times $0 < s < t$ and taking expectation, we get

$$C_\varepsilon(\Omega \setminus X_{[0,s]}, X_s) = \mathbf{E}_{X_{[s,t]}}[C_\varepsilon(\Omega \setminus X_{[0,t]}, X_t)]$$

Further...

For simplicity, consider

$$C_\varepsilon(\Omega, a) = \mathbf{E}^{\mu_\varepsilon}_{X^\varepsilon_{[0,t]}}[C_\varepsilon(\Omega \setminus X^\varepsilon_{[0,t]}, X^\varepsilon_t)]$$

Have 3 types of ε's:

- "coarseness" of X
- measure
- percolation scale

for the first 2, can coarsen space of curves and use $\mu_\varepsilon \rightharpoonup \mu'$

So really need

$$\text{“} \int C_\varepsilon \, d\mu_\varepsilon \to \int C_0 \, d\mu' \text{”}$$

Have no uniform convergence, instead, uniform (equi)continuity:

Restricted Uniform (Equi)continuity Lemma

Lemma

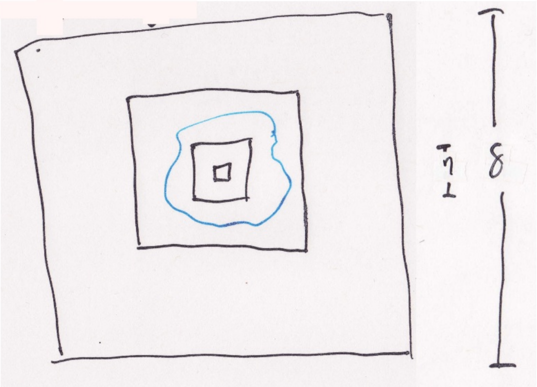
Given $\theta > 0$, $\exists \eta > 0$ *and there exists a set* Ψ, *such that*

$\forall \varepsilon$ *small enough* ($\varepsilon \ll \eta$), *for* $\gamma_1 \notin \Psi$, *and* $Dist(\gamma_1, \gamma_2) < \eta$:

1. $|C_\varepsilon(\Omega \setminus \gamma_1) - C_\varepsilon(\Omega \setminus \gamma_2)| < \theta$
2. $\mu_\varepsilon(\Psi) < \theta$

The same conclusion holds for μ'.

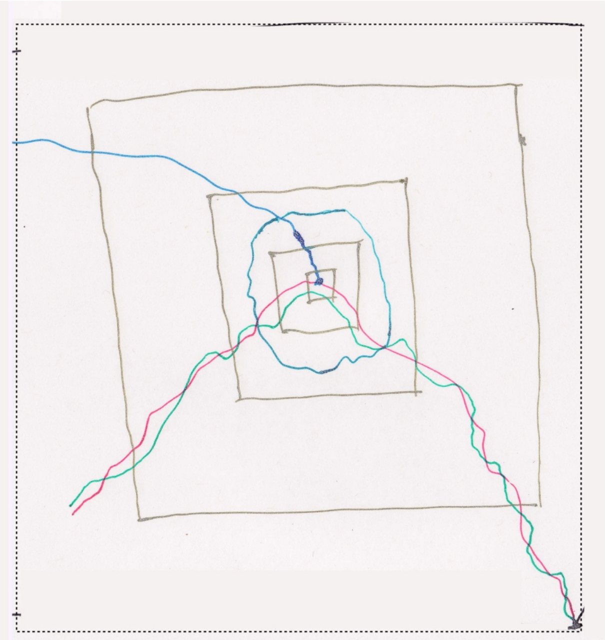
RSW



$\log(\delta/\eta)$ annuli
$\mathbf{P}(\exists \text{ ring}) \geq \alpha$ in each

$$\mathbf{P}(\nexists \text{ ring}) \leq (1-\alpha)^{\log(\delta/\eta)} \leq (\eta/\delta)^{\alpha}$$

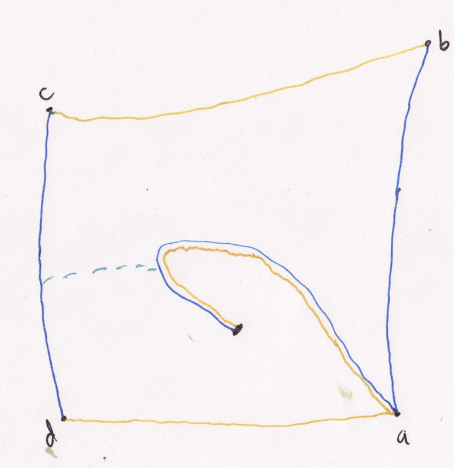
So...

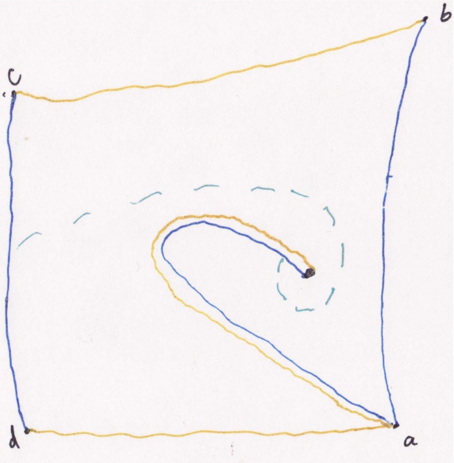


$\mathrm{dist}(\gamma_1, \gamma_2) < \eta$,
w.p. $\to 1$
as
$\eta/\delta \to 0$
crossing for
$\Omega \setminus \gamma_2$
is also crossing for
$\Omega \setminus \gamma_2$

However...

Curves are 2–sided: Starting from a, the blue side is on the right:

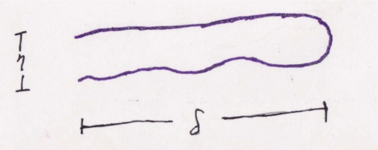




"Counterexample"

No Doublingback

δ/η–doublingback:



$$\mathbf{P}(\exists\ \delta/\eta\text{–doublingback}) \leq e^{-c(\delta/\eta)}$$

for $\varepsilon \ll \eta$ (by RSW and BK)

***note scale invariance: only depends on δ/η

Multiscale version:

$\log \delta/\theta$ scales, κ–υ bad box if in $> 1 - \upsilon$ fraction of scales have κ–doublingback

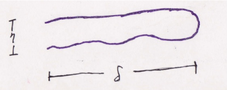
$$\mathbf{P}(\exists\ \kappa\text{–}\upsilon \text{ bad box}) \leq \frac{C}{\theta^2}\left(\frac{\theta}{\delta}\right)^{\alpha}$$

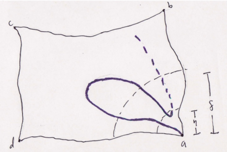
Quantification

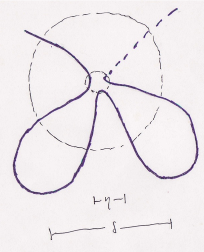
- Many scales:

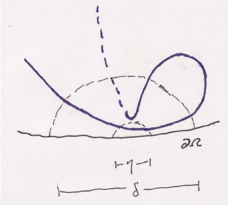
$$\eta \ll \vartheta' \lesssim \vartheta \ll \delta_3 \ll \delta_2 \ll \delta_{3/2} \ll \delta_1 \ll \Delta_4 \ll \tilde{\Delta} \lesssim \Delta \ll \Delta_1 \ll 1$$

- The set Ψ:



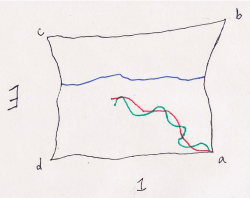




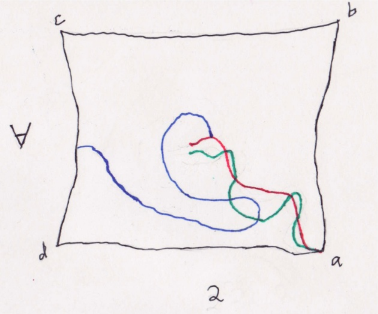


3 Cases

Sufficient to show w.h.p. crossing for $\Omega \setminus \gamma_1 \to$ crossing for $\Omega \setminus \gamma_2$: 3 cases (which disjointly partition the percolation configuration space)

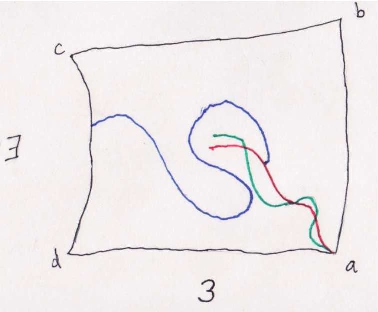


$\exists$ crossing independent of γ_1 or γ_2



all crossings land on γ_1 and pass through γ_2

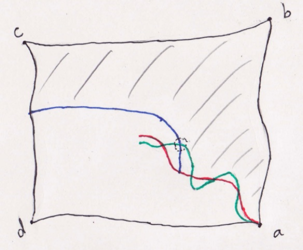
$\rightsquigarrow$
w.h.p.



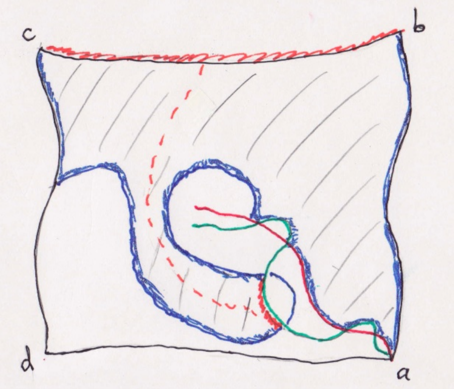
not in case 1 and $\exists$ crossing which lands on γ_1 and does *not* pass through γ_2

Reduction to Case 3

If in case 2 (all crossings land on γ_1 and pass through γ_2), then



either blue crossing for $\Omega \setminus \gamma_2$

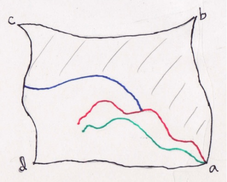


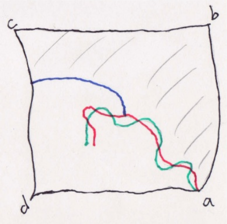
or case 2 with yellow $\leftrightarrow$ blue, $2 \leftrightarrow 1$

In case 2, sufficient to RSW continue blue crossing to γ_2

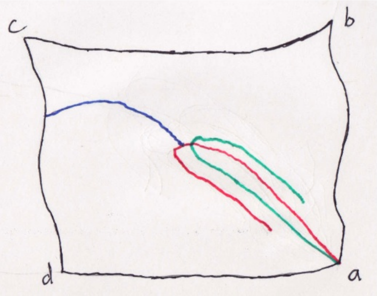
Reduction to Highest Crossing

If in case 2, then highest crossing (in the domain $\Omega \setminus \gamma_1$) satisfies conditions of case 2 (lands on γ_1 but does *not* pass through γ_2):





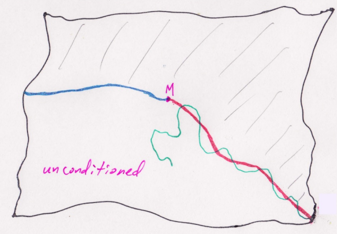
However, with doublingback, the orientation of γ_2 may change in such a way that a higher crossing will cross the *yellow* side of γ_2.



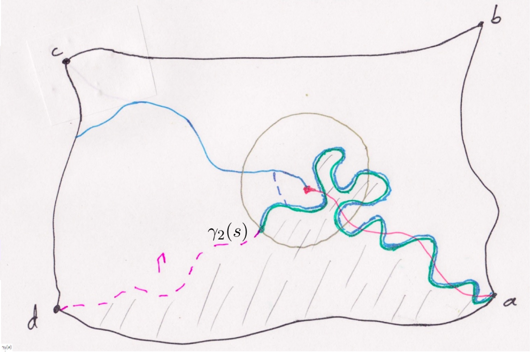
This can be handled. To illustrate this sort of argument...

Correct Topological Picture

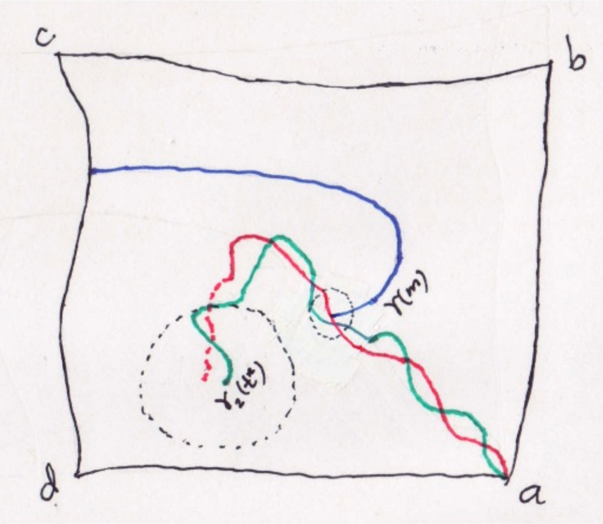
Suppose in case 3 and have selected the highest crossing:



If such a $\Gamma : \gamma_2(s) \rightsquigarrow d$ (does not cross blue crossing or $\gamma_2([0,s])$) exists, then any RSW continuation inside ball guaranteed to hit the blue side of γ_2:



The Point $\gamma(t^*)$

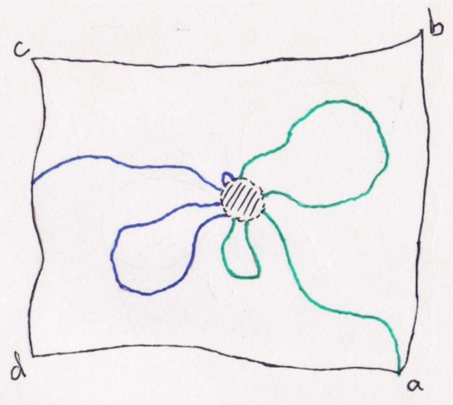


- RSW $\to \gamma_2(t^*)$ far from blue crossing
- No doublingback $\to$ $\gamma_2(t_*)$ is far from $\gamma_2([0,s])$
- Suffices to show $\exists\Gamma : \gamma_2(t^*) \to d$ avoiding blue crossing and $\gamma_2([0,s])$

Multiply Connected Domains

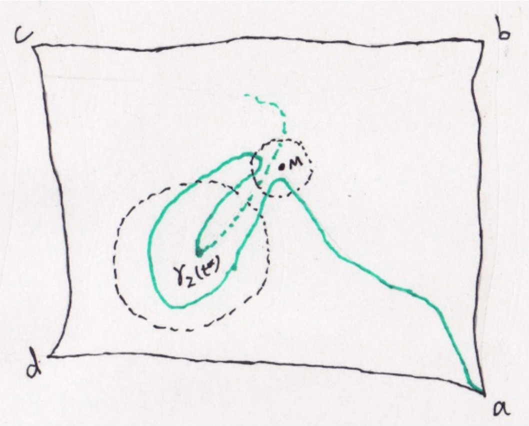
Basically, need to show w.h.p., $\gamma_2(t^*) \in C_{F_g}(d)$, where

$$F_g = \Omega \setminus [\gamma_2([0,m]) \cup B_\eta(M) \cup \text{blue crossing}]$$



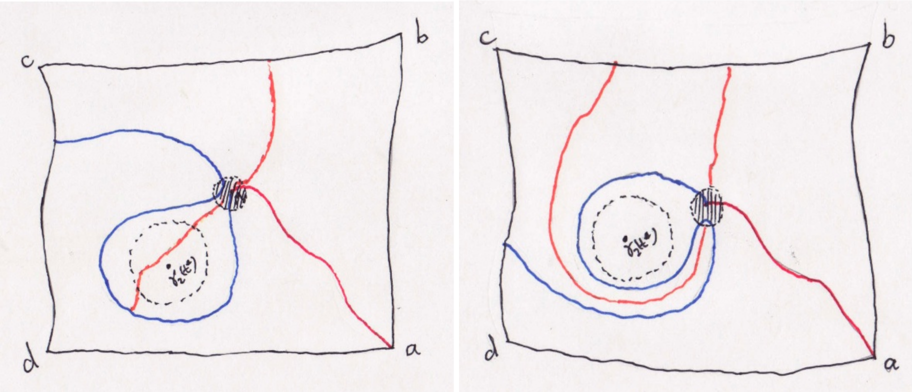
Note F_g has small components and $C_{F_g}(b)$ & $C_{F_g}(d)$

Small Components: Green Pods



Being inside a green pod means γ_2 makes a triple visit to $B_\eta(M)$.

Small Components: Blue Pods

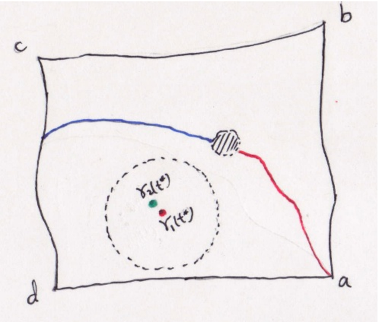


Highest crossing means being inside a blue pod implies 5 long arms emanating from $B_\eta(M)$, which has vanishing probability, since $M(\gamma) < 2$.

Large Components

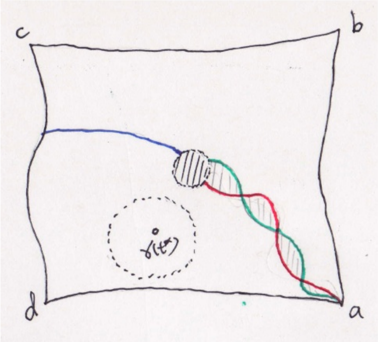
Remains to show $\gamma_2(t^*) \notin C_{F_g}(b)$. Now assume no small components:

Clear that $\gamma_1(t^*) \in C_{F_r}(d)$ so $\gamma_2(t^*) \in C_{F_r}(d)$ also



Conclude $\gamma_2(t^*) \in C_{F_g}(d)$

$\gamma_2(t^*) \in C_{F_r \cap F_g}(b)$ or $\gamma_2(t^*) \in C_{F_r \cap F_g}(d)$



Continuation of Crossing