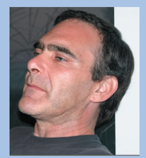
UCLA Department of Mathematics

UCLA Probability Seminar
February 15, 2006

Cardy's Formula for Certain Models of the Bond Triangular Type

Joint work with L. Chayes 

Talk Outline

I Background & Smirnov's Proof

II Triangular Bond Model

III Model Under Consideration

IV Path Designates

V Color Symmetry Without Conditioning

VI Color Symmetry Under Conditioning

VII Crossing Probabilities

VIII Summary of Technical Difficulties

IX Loop Erasure

Work of Smirnov takes place on the triangular site lattice, equivalently hexagon tiling of $\mathbb{C}$.

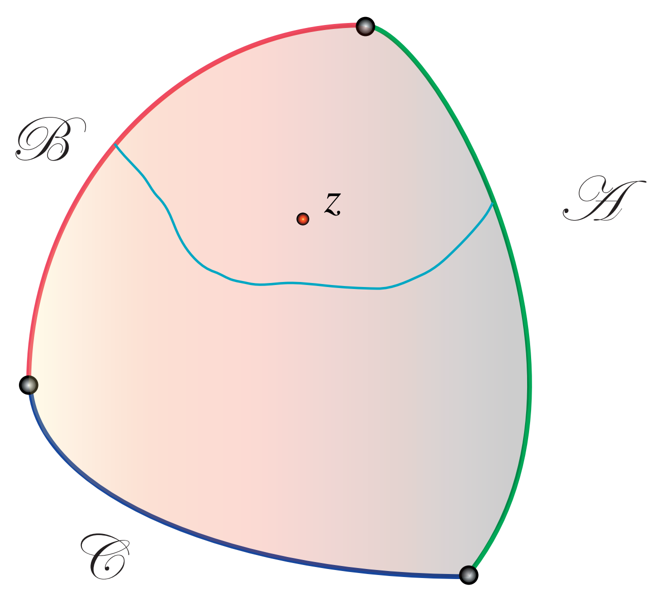
 w/prob p.

 w/prob $(1-p)$.

Critical at $p_c = 1/2$

Central Practical Goal

U(z):



$$u(z) = \mathrm{P}\,(\mathbf{U}(z)),$$

As lattice spacing tends to zero, $u(z)$ converges to a harmonic function.

Uniquely specified by boundary and analyticity conditions, hence conformally invariant.

Key Ideas

I. Harmonic Triples (120 degree symmetries)

- $u + \frac{i}{\sqrt{3}}(v - w)$, etc. are analytic functions.

- 120 degree Cauchy-Riemann type equations like

$$D_{\hat{s}} u = D_{(\tau \hat{s})} v \quad ; \quad \tau = \exp(\tfrac{2\pi i}{3})$$

- Equilateral triangle: *u, v* and *w* are linear and do satisfy Cardy's formula.

Enough to show on an arbitrary domain *u, v, w* satisfy the same boundary/derivative conditions as on the equilateral triangle (solution to the same conformally invariant problem).

II. Lattice Functions

Boundary/derivative conditions seen from the lattice functions.

- Boundary conditions are easy and lattice independent.
- Main difficulty: Cauchy-Riemann Equations.

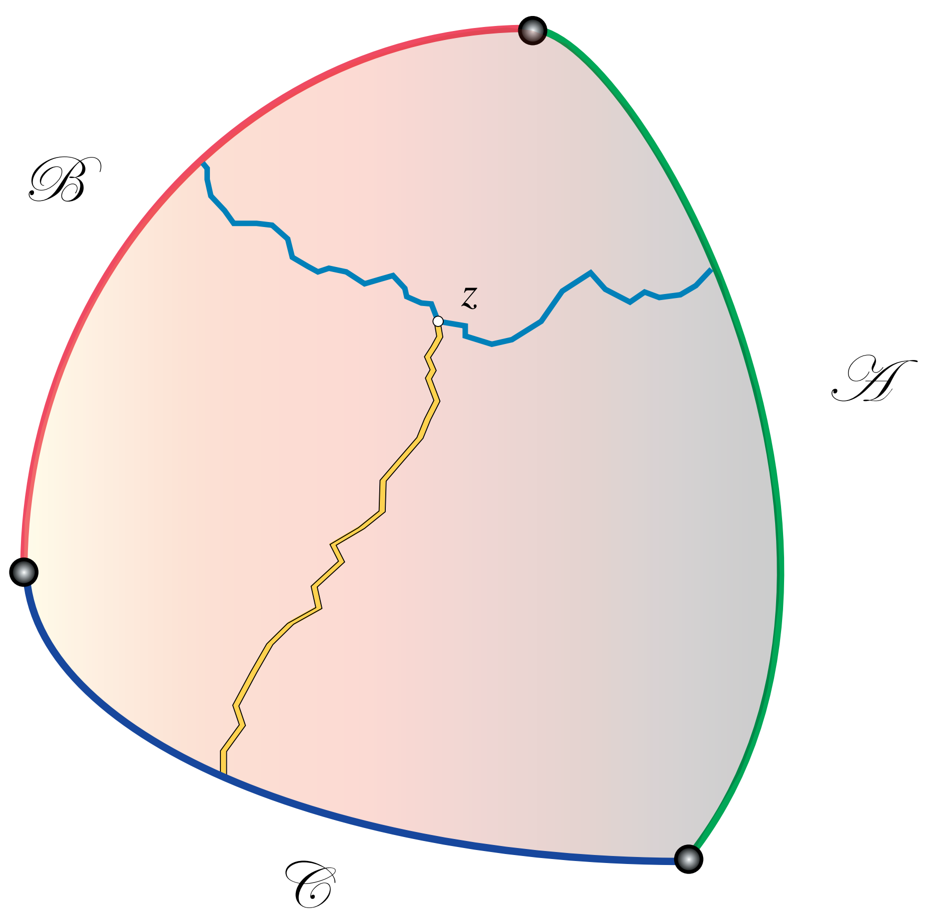
Discrete Derivatives and Color Switching

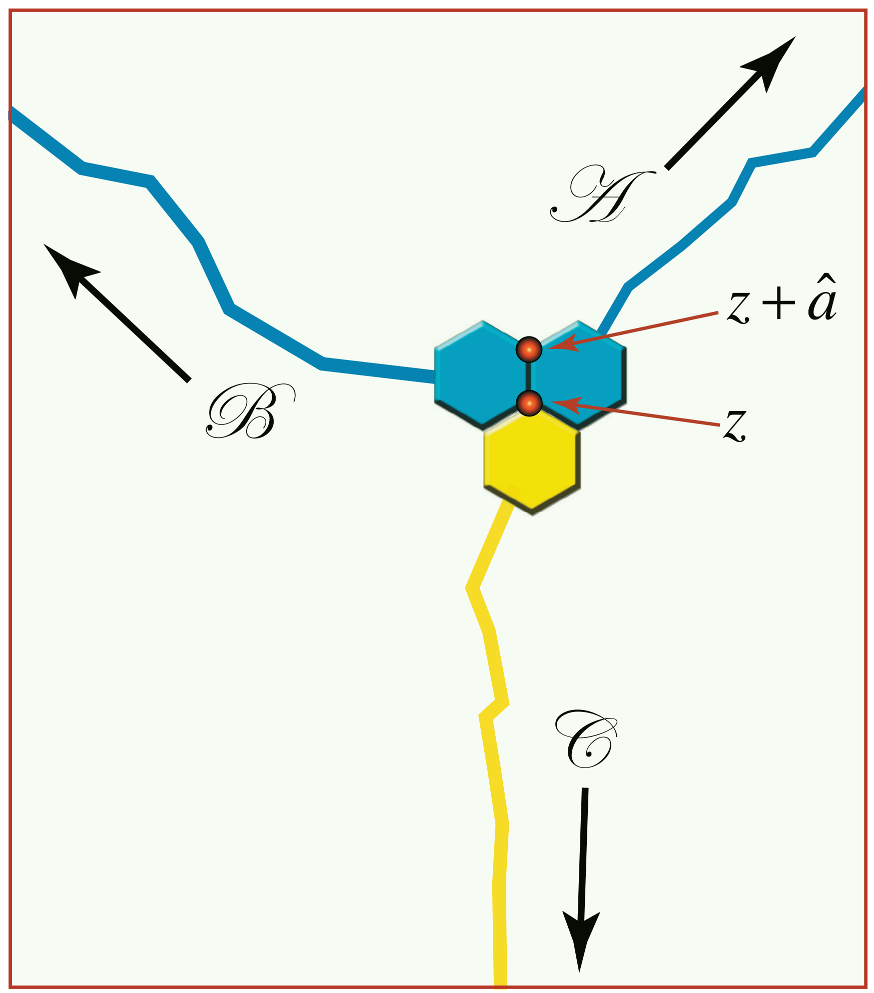
- The discrete derivative is given by

$$u(z+\hat{a})-u(z)$$

which is seen to equal

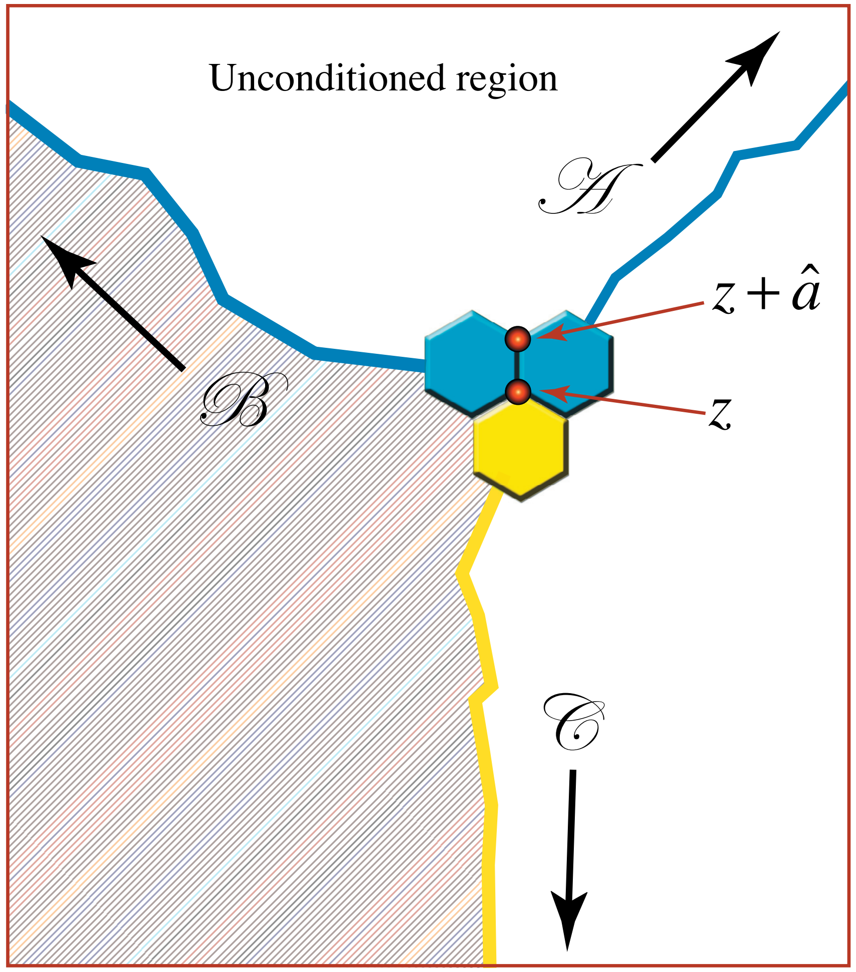
$$\mathrm{P}\,[\mathbf{U}(z+\hat{a})\setminus\mathbf{U}(z)] - \mathrm{P}\,[\mathbf{U}(z)\setminus\mathbf{U}(z+\hat{a})] = \mathbf{U}_a^{+}(\mathrm{z}) - \mathbf{U}_a^{-}(\mathrm{z})$$

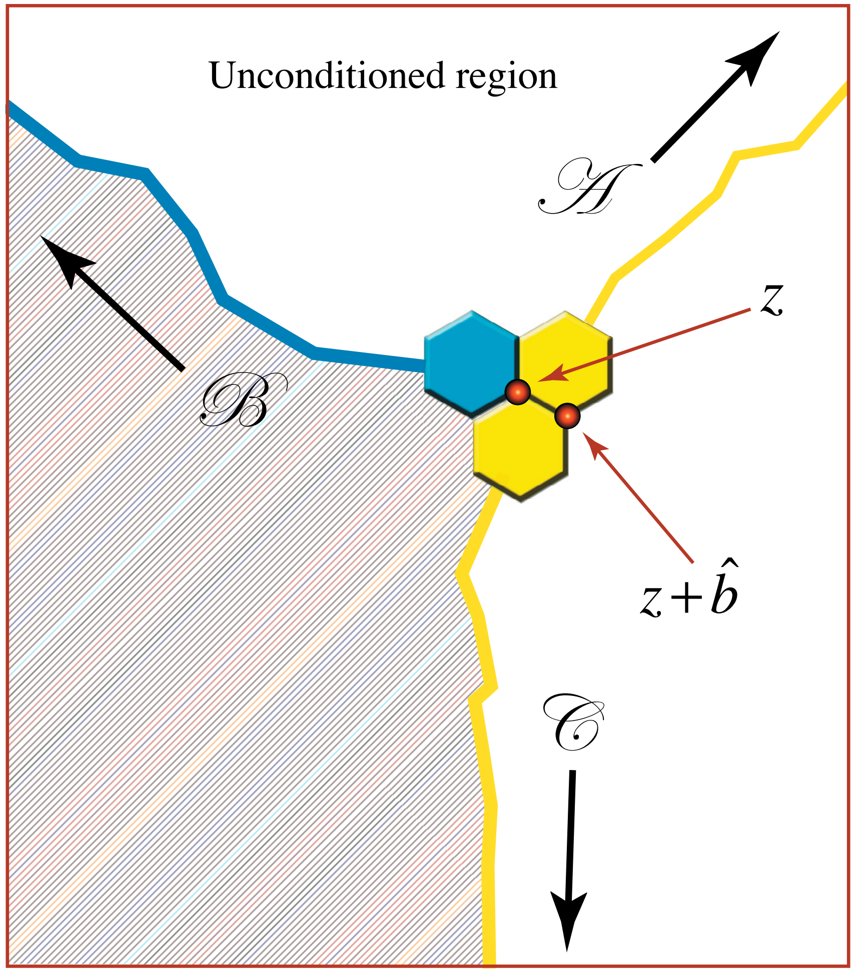




The CR relations: let $\hat{a}$ & $\hat{b}$ be two lattice vectors as shown, then

$$\mathbf{U}_a^+ = W_b^+$$



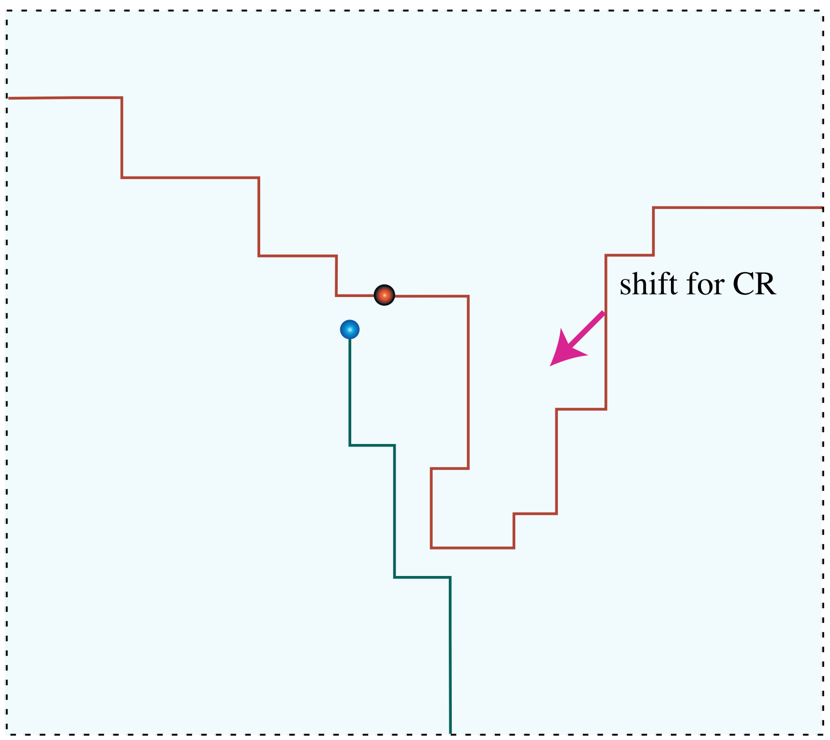


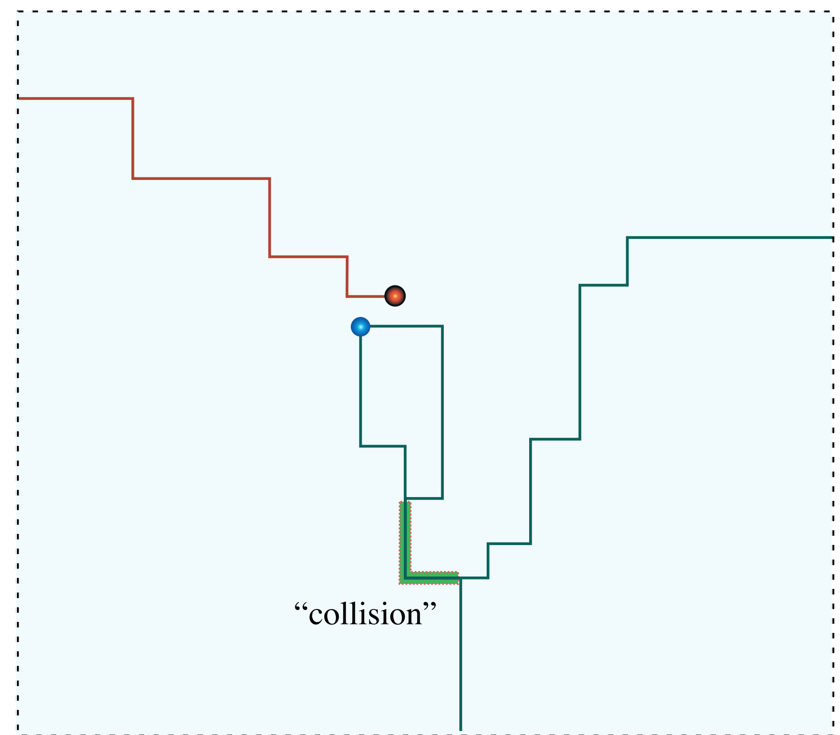
i.e. the probabilities of the two “CR–pieces” are the same.

This together with some analysis is enough to push through a proof.

Difficulties With Other Lattices

It is a miracle of the triangular site lattice that these innocuous looking CR relations hold without apology. E.g., on the square lattice:



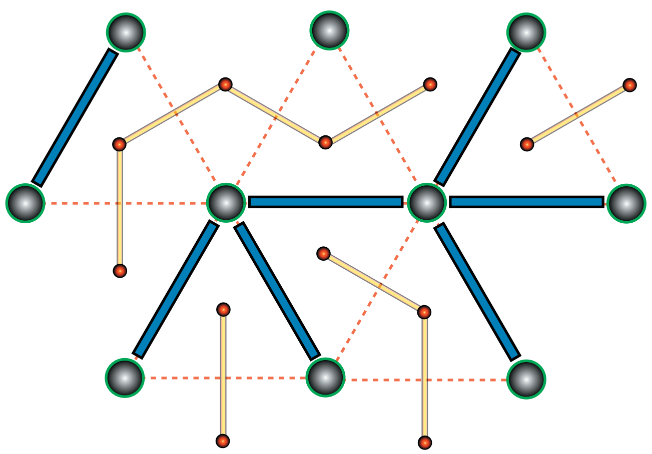


Here “collision” problems occur when trying to switch colors.

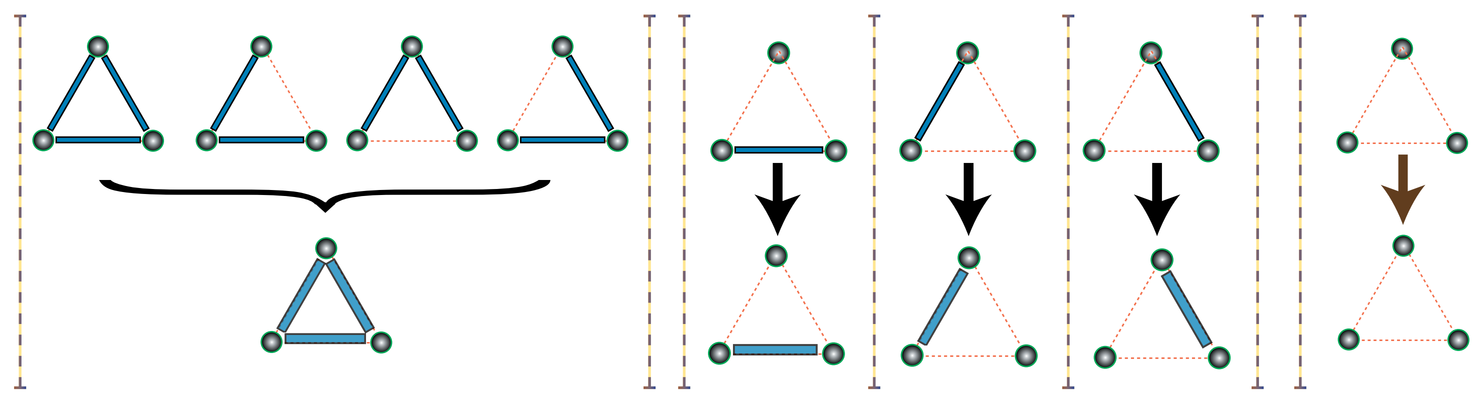
L. Chayes and H.K. Lei, *Random Cluster Models on the Triangular Lattice*, to appear in J. Statist. Phys

Model based on triangular lattice *bond* percolation problem.

(1) Bonds independently *blue*: p / *not–blue*: $(1–p)$.



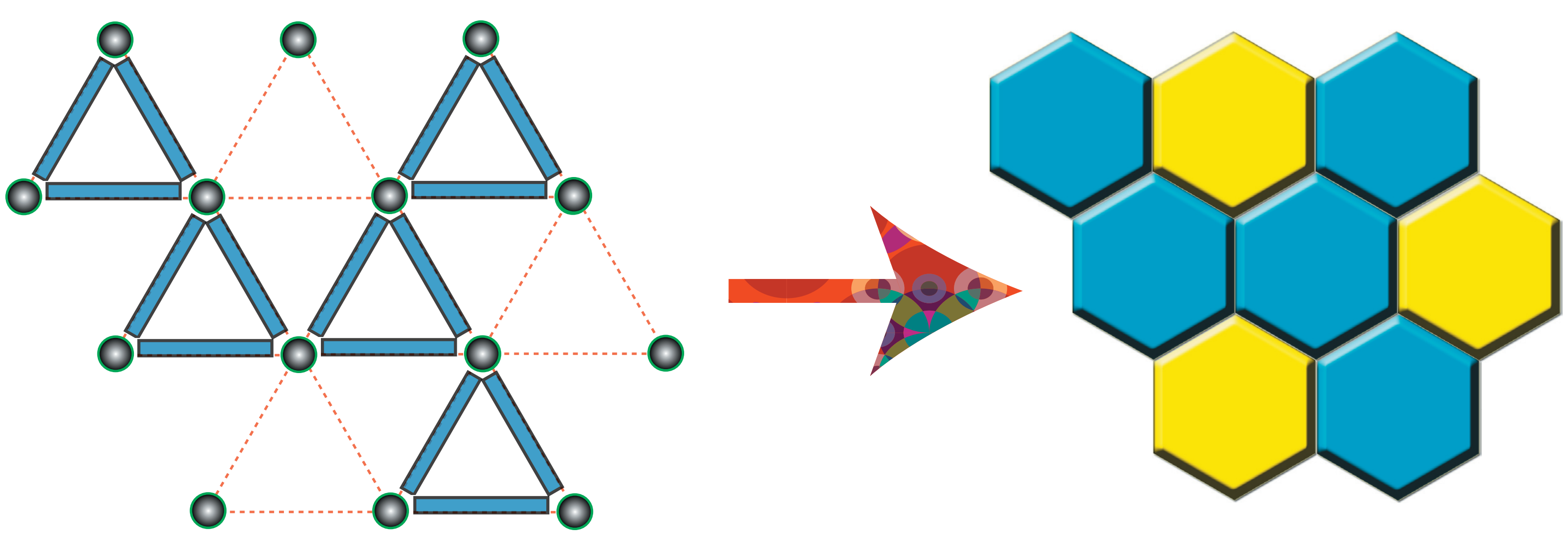
(2) On each up–pointing triangle – 8 configurations – may as well reduce vis–á–vis connectivity properties:



a s e

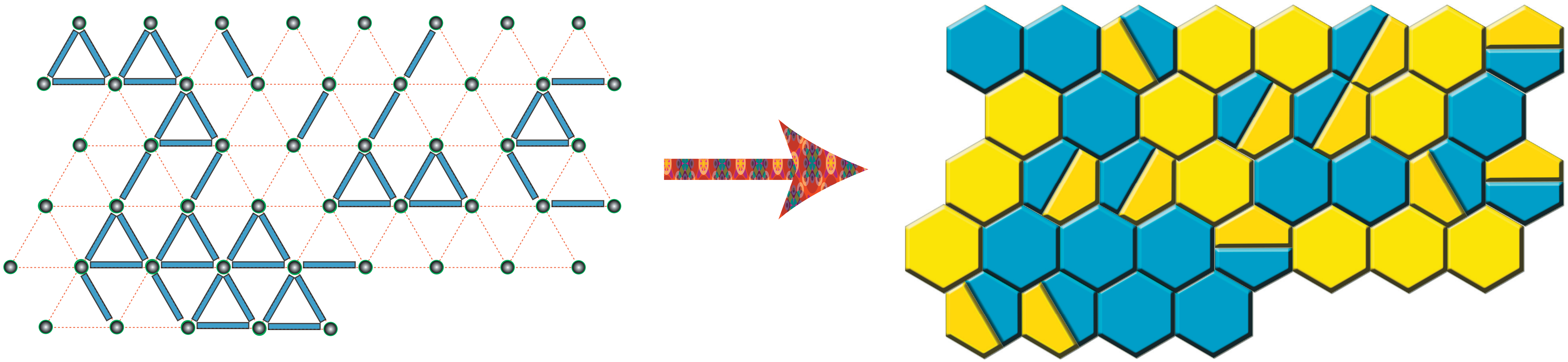
(3) Now a locally correlated percolation problem. Self–dual (via ★–▲ transformation) at $a = e$. And critical – $ae > 2s^2$ [CL].

(4) Note, $s = 0$ (i.e. $a+e = 1$) is exactly triangular site percolation problem:



<u>Claim</u>:

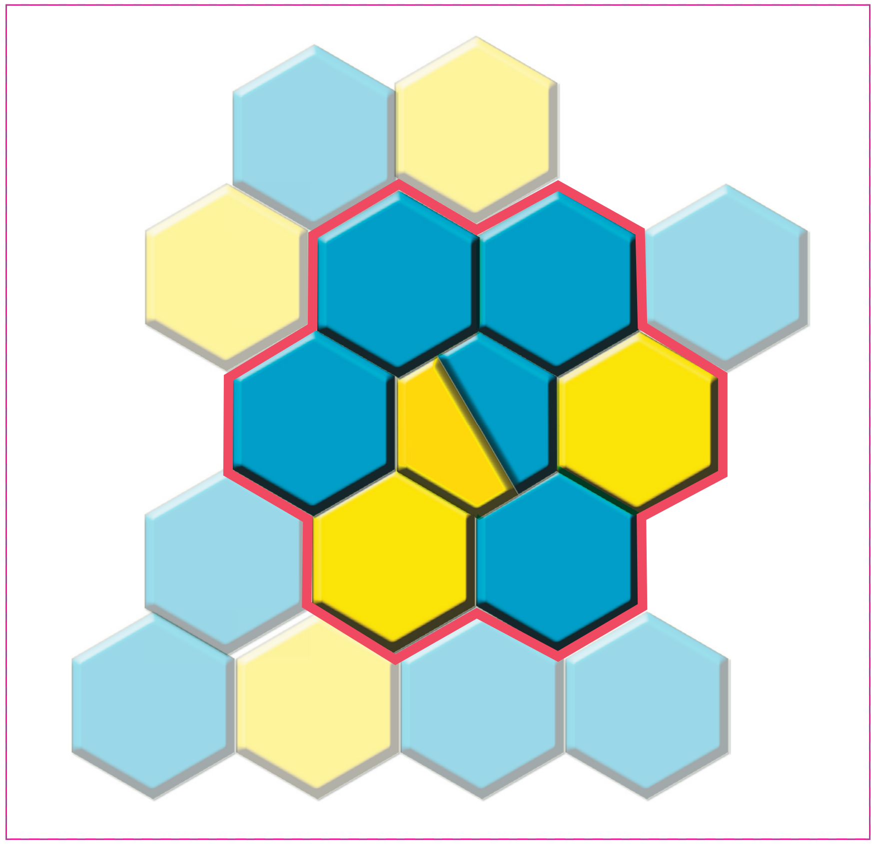
Add in single bond events (probability $s \neq 0$) $\Longleftrightarrow$ Introducing split hexagons into the problem.



Remark:

Unfortunately, full triangular bond lattice problem too hard. Need (local) correlations.

Geometric Setup



Objects of consideration:

flowers, *irises*, *petals*.

Tile the domain with hexagons, some of which are designated to be irises, such that flowers are disjoint.

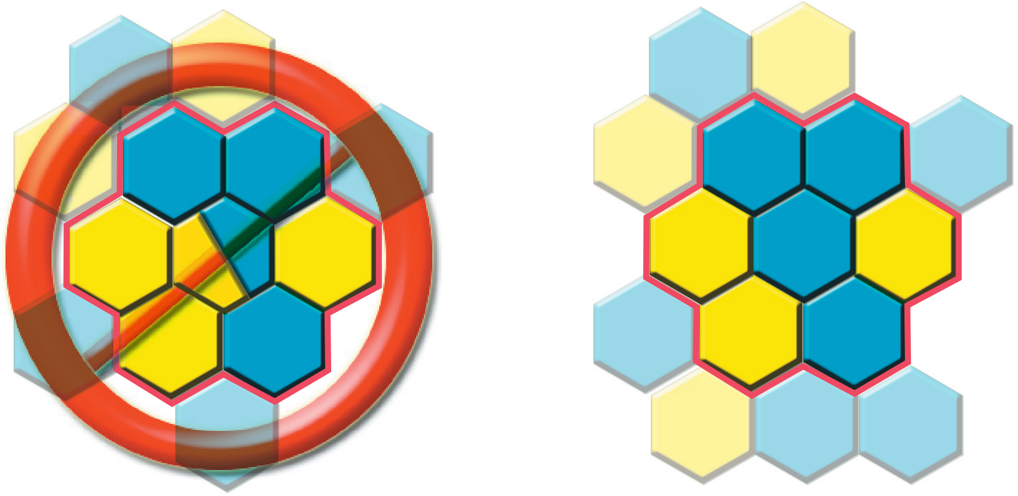
Rules

- Non-irises can only be blue or yellow, with equal probability.

- Iris can be blue, yellow, or mixed with probabilities a, a ($a \equiv e$) and s, respectively (so $2a+3s = 1$),

EXCEPT

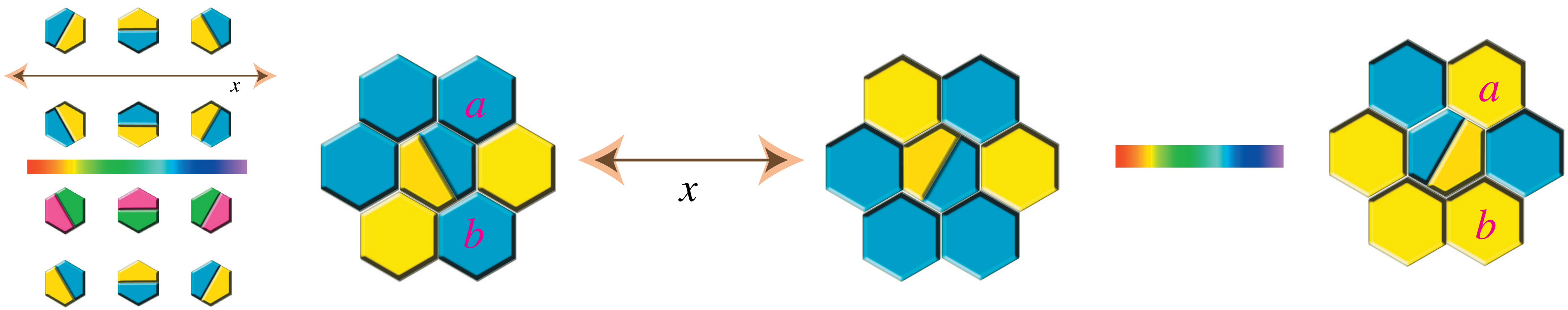
- In *triggering* situations, where the iris ceases to be an iris. Note this introduces local correlations.



- Disjoint flowers are independent.

Flowers

Hope to restore some color symmetry flower by flower. Indication this may work:



Reflection/color reversal gives 1-1 and onto map between the colors.

Not good enough. Need triggering.

Triggering

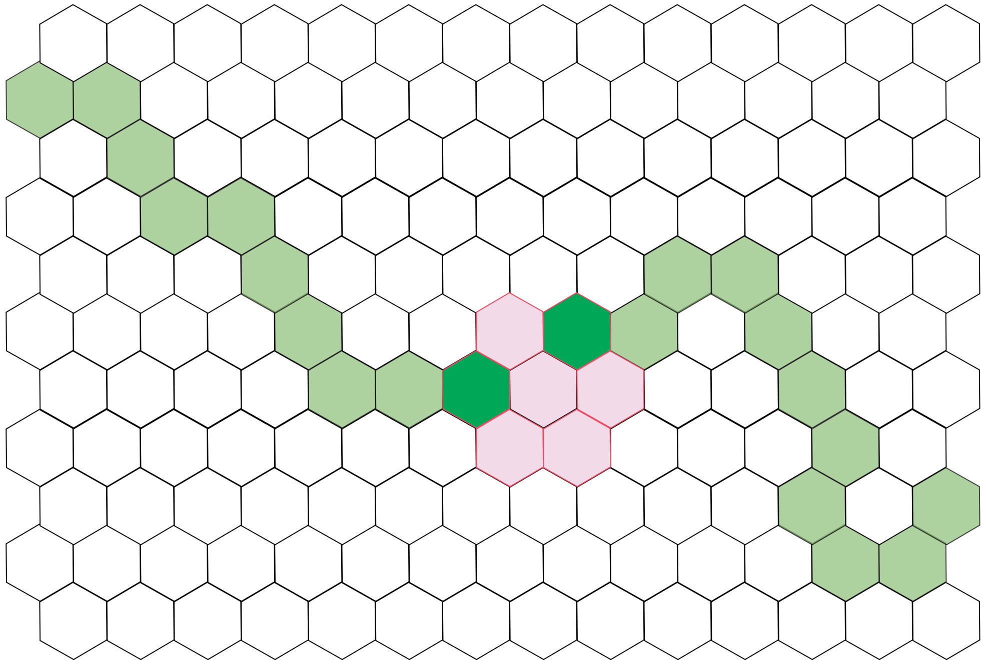
- $\frac{3}{16}$ of all possible configurations on a flower.
- The $$price$$ we pay:
  - Lose FKG in general (but still have it for path events)
  - A host of other difficulties to follow.

- On the bright side, these deviations due to triggering reassure us that our model is indeed different from the triangular site model and cannot be viewed as an “easy” limit of it.

We have no microscopic color symmetry, so need to consider paths “modulo flowers”.

Path Designates

A path going through a flower enters at some *entrance petal* and exits at some *exit petal*.



A *path designate* specifies the path outside of flowers but only specifies the entrance/exit petals for flowers - in order.

Given a path designate $\mathscr{P}$, we let $\mathscr{P}_B$ denote the event that there is a *realization* of $\mathscr{P}$ in blue. Similar for $\mathscr{P}_Y$.

We generalize these notions in the obvious way to the case of multiple flowers and multiple visits to a single flower.

As a collections of paths, not useful as a partition of the configuration space.

As geometric objects, problematic since not specific enough.

BUT ESSENTIAL FOR OBTAINING COLOR SYMMETRY.

For our purposes, we do not let a path designate start on an iris.

Lemma 1

Let **r** and **r′** denote non-iris hexagons). Then the probability of a monochrome path between **r** and **r′** is the same in blue as it is in yellow.

We prove the result flower by flower and then concatenate:

Let $\mathscr{F}$ denote a flower and let $\mathscr{D}$ denote a collection of petals of $\mathscr{F}$. Let $T^B_{\mathscr{D}}$ denote the event that all the petals in $\mathscr{D}$ are blue and that they are blue connected in the flower. Let $T^Y_{\mathscr{D}}$ denote a similar event in yellow. Then

$$\text{For all } \mathscr{D}, \qquad \mathbb{P}(T^B_{\mathscr{D}}) = \mathbb{P}(T^Y_{\mathscr{D}})$$

We will in fact need the multiset version of Lemma 1.1 (i.e. $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$ and $T^Y_{\mathscr{D}_1 \ldots \mathscr{D}_k}$) but due to limitations of flower size, these cases do not present any additional difficulty.

Given this local result, the lemma follows by an inclusion-exclusion argument.

Lemma 1 + periodic floral arrangement + $ae \geq 2s^2$ can be used to establish typical critical behavior:

- No percolation of yellow or blue.
- Rings in annuli (with uniform probability) @ all scales.
- Power law bounds on connectivities.

But for us, this is just the beginning. We must face up to problem of color symmetry for transmissions in presence of *conditioned paths*.

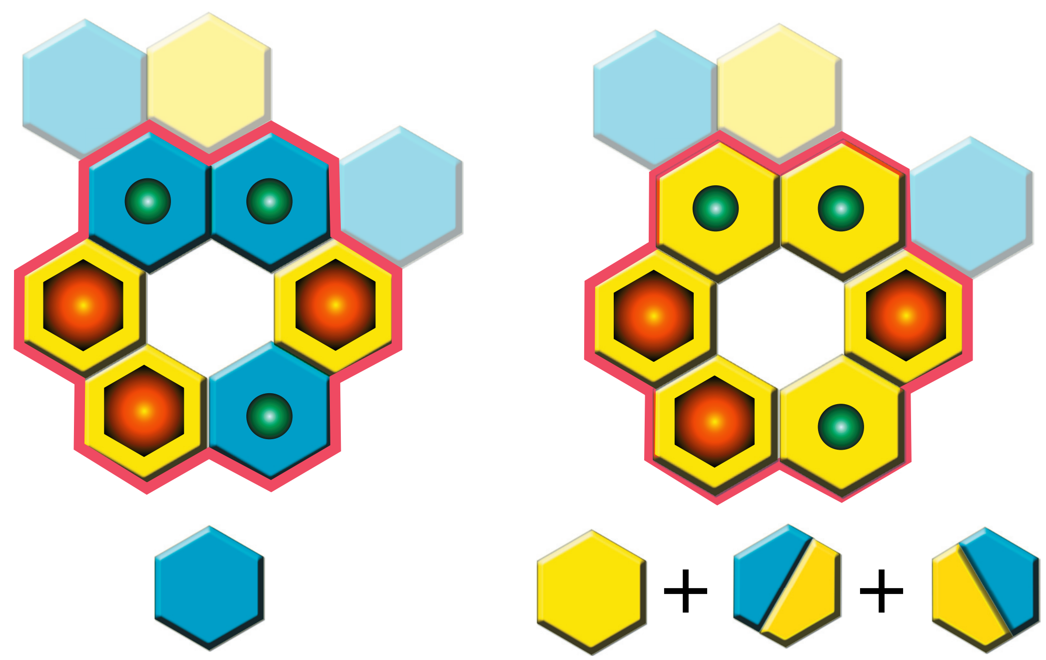
For CR need to change color in presence of *conditioning*.

PROBLEM

Example:

Conditioned Sites

Transmission Ports



$\frac{1}{2}$ (Trigger) $\qquad a + 2s$

SOLUTION

Rethink the meaning of disjoint

DEUS EX MACHINA

When blue at disadvantage, allow blue conditioned petals to be shared with some probability.

When blue at advantage, forbid from touching blue petals used by the conditioned set.

PREVIOUS EXAMPLE

Always fine

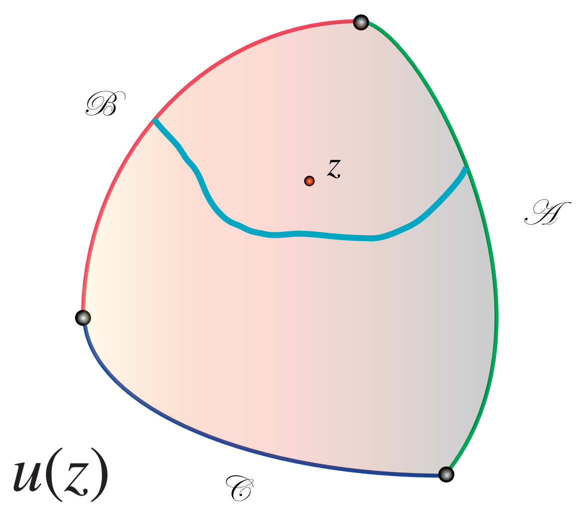
with probability

$$\frac{s}{2(a+2s)}.$$

Lemma 2

There exists a set of random variables and corresponding *-rules (laws for random variables) such that the conclusion of lemma 1 holds in the presence of conditioning.

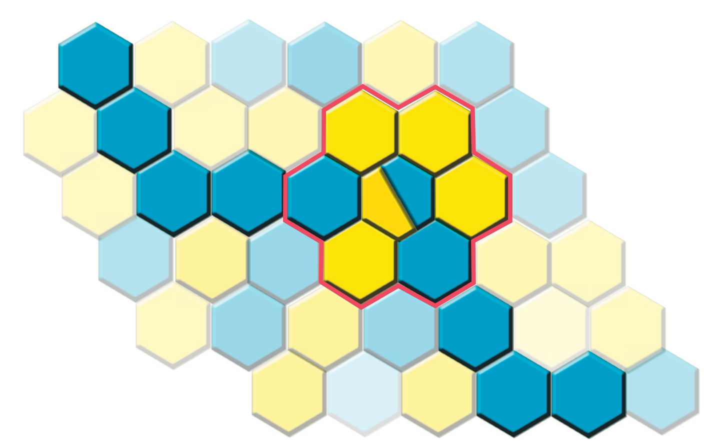
What does all this mean for our functions u_N, v_N and w_N? (N denotes lattice spacing of N^{-1})



STRATEGY

I. Prove what we want for *-versions of the functions: u^*_N, v^*_N and w^*_N.

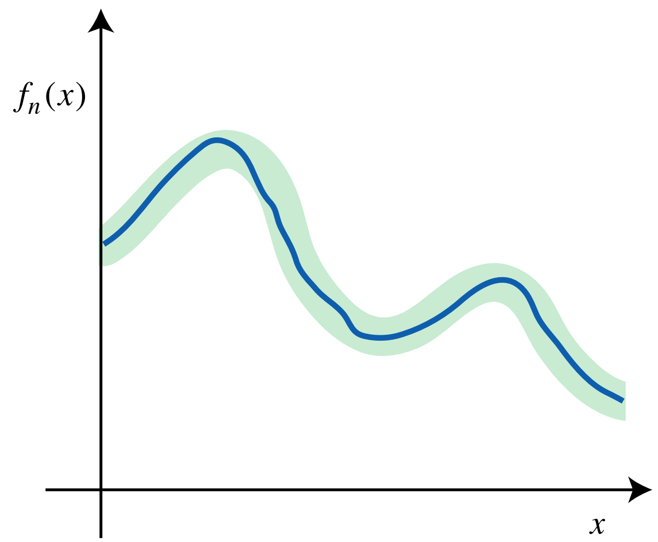
II. Then do some analysis to show e.g. $|u_N - u^*_N| \to 0$.

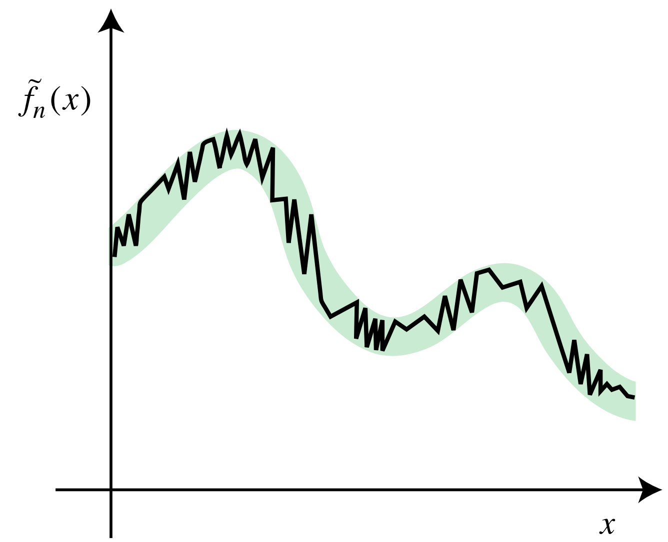


$u^*_N \notin \{0,1\}$

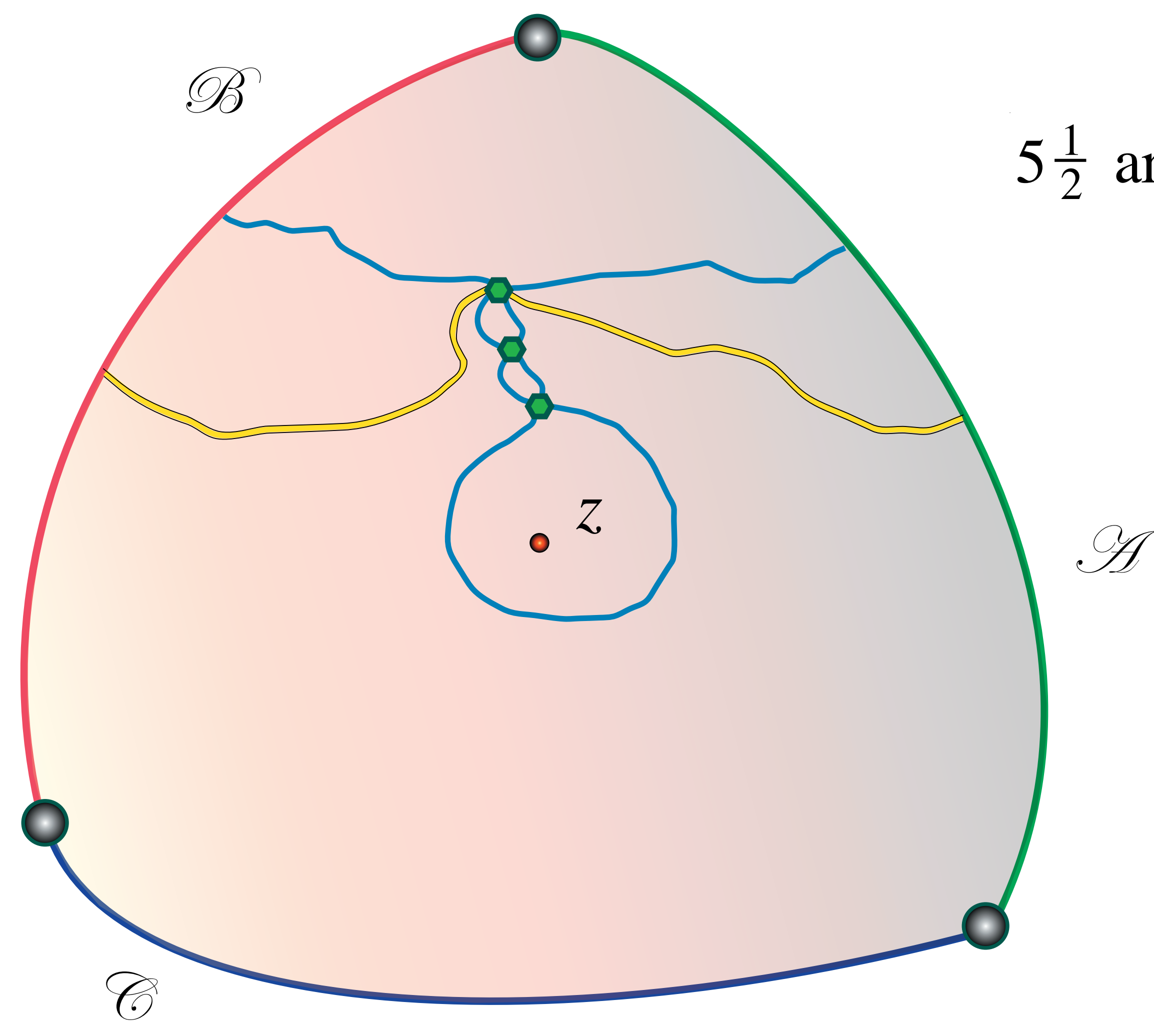
color switching lemma + a contour argument + more gives (I).

General picture of (II) is





$5\frac{1}{2}$ arm event



- Path satisfying "event" only come near z with vanishingly small probability.

- Path of a configuration in $\mathcal{U}^*_N(z) \Delta \mathcal{U}_N(z)$ not close to z lead to "five and a half" arms, which occur with vanishingly small probability.

The $$price$$ of color symmetry:

I. FKG inequality and RSW lemmas.

- FKG was ostensibly difficult, but the assumption of $a^2 \geq 2s^2$ and the result in [CL] made it easy.
- For RSW, among other difficulties, had to actually read Kesten's book.
- Tragedy of RSW: lost rights to arbitrary floral arrangements.

II. Arms and Exponents.

- A five and a half arm argument, along with a three arm argument in the complement of a line segment was needed to show equivalence of Carleson-Cardy functions.
- Due to local correlations, standard KvB or Reimer's inequality does not apply, needed old fashioned conditioning argument.

III. Full Flower vs. "Used" Flower.

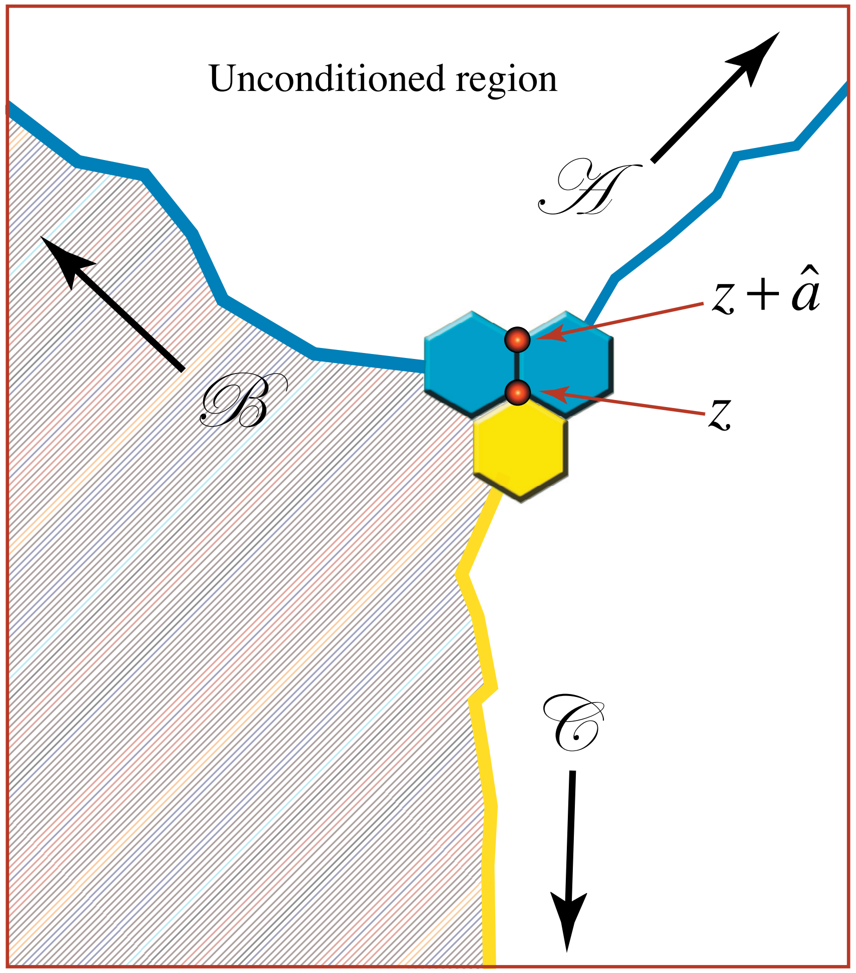
- This was needed in the conditioning argument in II.
- Seemingly "obvious", but involved meticulous and systematic consideration of all possibilities.

IV. The Iris in Cauchy-Riemann Switch.

- No sensible mechanism to have path designate start @ iris. CR–relations require effort.

V. Producing the Lowest Path for Conditioning (loop erasure).

Had to condition on “lowest” paths, need to ensure some paths are self-avoiding/non-self-touching.



PROBLEM

The *-paths are NOT self-avoiding/non-self-touching.

QUICK CURE

Take a geometric path and delete all loops.

COMPLICATIONS

I. Must keep loops that “capture” z.

II. Random variables may cause unwanted “dumping” after deletion of loops.

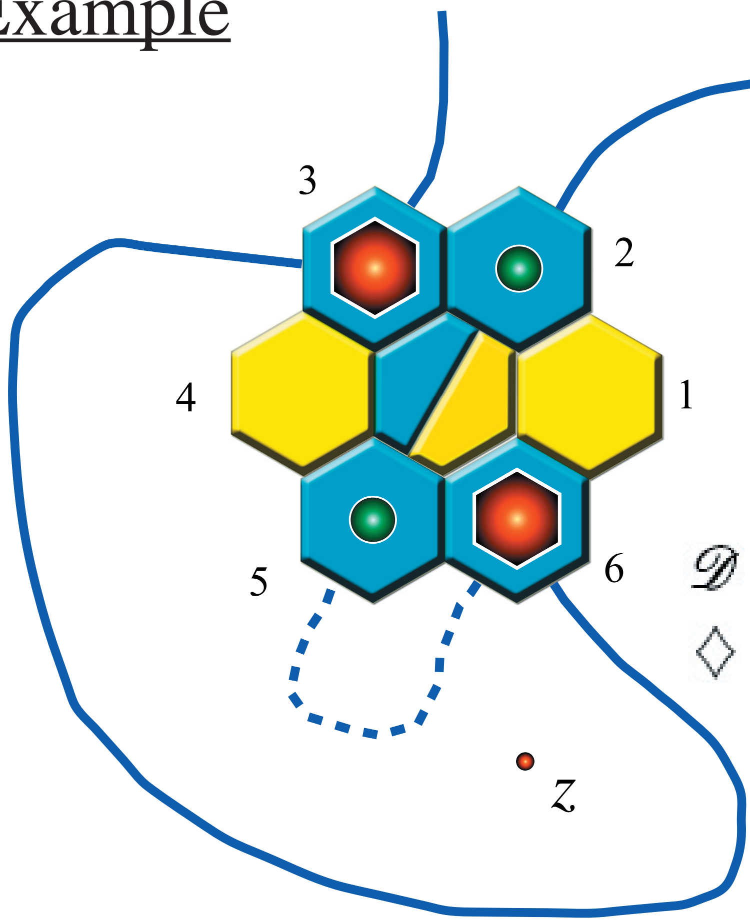
When is a path *-good?

- First pass through flower is “free”.
- Pass N through flower takes petals used by first N-1 passes as conditioned set.

Must receive “permission” from all relevant random variables!

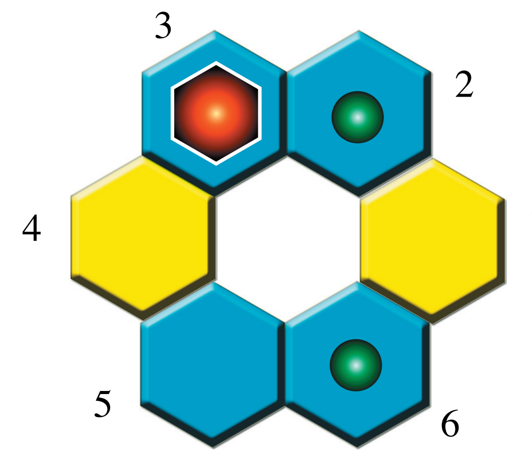
Example

In fact, the statement that the fully reduced version of Γ (i.e. all loops erased except for the one necessary for “capture” of *z*) satisfies the event is false:



$\mathscr{D}$ = {5,2}

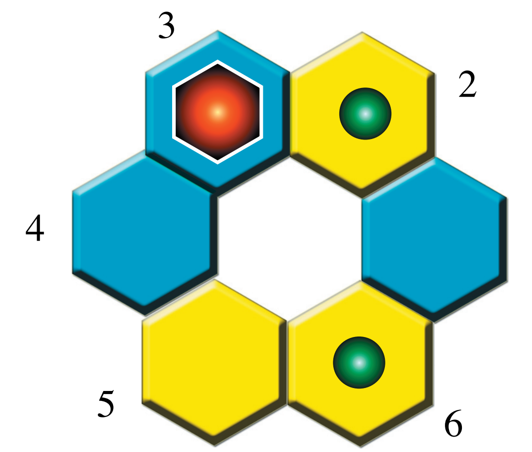
$\diamondsuit$ = {3,6}



$\mathscr{D}$ = {2,6}

$\diamondsuit$ = {3}

Transmission: $(a + 2s)$



<u>Trigger</u>

Transmission: $\frac{1}{2}$

THE TRUTH

Can reduce half the path (from boundary to first bottleneck of loop with z in interior).

This is all we need.

(I) Wrap–Up. After much work, result is that lattice functions u_N, v_N & w_N for *this* model converge to the "Cardy–Carleson" functions.

I.e. the same result as for triangle site lattice model.

Pretty much a complete proof that the continuum limits of both systems are exactly the same; Reasonable and fairly robust statement of *universality*.

Central dogma for theory of critical phenomona since the 1960's.

(II) Limitations

(a) Not a standard (well known) percolation model.

(b) Within context of model, didn't get most complete result.

Although model does indeed have parameters –"some generality".

(c) Aside from some practical (and technical) considerations, did *not* learn much about the nature of and convergence to continuum limit – beyond what was already known.