Viscosity Solution for Degenerate Diffusion with a Drift

(Joint work with I. Kim)

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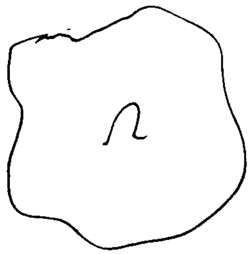
May 4, 2010

Elliptic Eqns and the Maximum Principle



$$Lu = a^{ij} D_{ij} u + b^i D_i u, \qquad a^{ij} = a^{ji}$$

elliptic if $a^{ij}(x) > 0$ — max. principle

unif. elliptic — strong max. principle (no nontrivial int. max)

$(u, v \in C^2(\Omega) \cap C^0(\bar{\Omega}))$.

Max Principle

$L(u-v) \geq 0$, then $\max(u-v)$ on $\partial\Omega$.

Hence

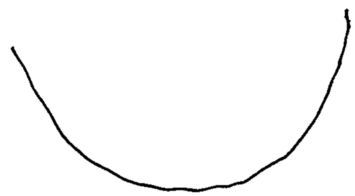
Comparison Principle (and uniqueness)

If 1). $u \leq v$ on $\partial\Omega$

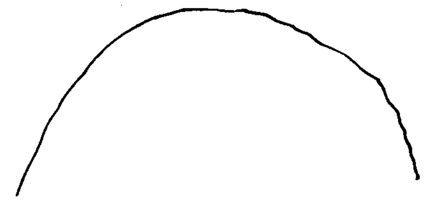
2). $L(u-v) \geq 0$,

then $u \leq v$ in Ω.

Subsolution and Supersolution



$Lu \geq 0$
(subsolution)



$Lu \leq 0$
(supersolution)

Idea Define e.g. viscosity subsolution by this property:

u is viscosity subsoln if $\forall \varphi \in C^2(\Omega) \cap C(\bar{\Omega})$

(see Crandall-Ishii-Lions)

s.t. $(u-\varphi)$ has max at x_0

$(u-\varphi=0, \ \nabla(u-\varphi)=0 \Rightarrow L(u-\varphi) \leq 0)$

then

$$0 \leq (\leq Lu) \leq L\varphi$$

For us, parabolic eqns: $Lu - u_t = 0$. get $\varphi_t \leq L\varphi$

Porous Medium Equation

(density) $\rho_t = \Delta \rho^m = \nabla \cdot (\nabla \rho^m)$ (divergence form)

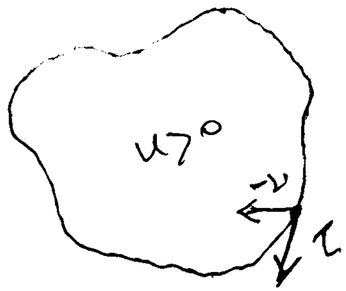
$$u = \frac{m}{m-1} \rho^{m-1}$$

(pressure) $u_t = (m-1) u \Delta u + |Du|^2$

- If $0 < \varepsilon < u_0$, unif. parabolic theory $\Rightarrow \exists!$ soln $0 < \varepsilon < u(x,t)$.
- Otherwise, degenerate $\longrightarrow$ free boundary

(u small, $u_t \approx |Du|^2$: preservation of 1). cpct support
2) positivity)

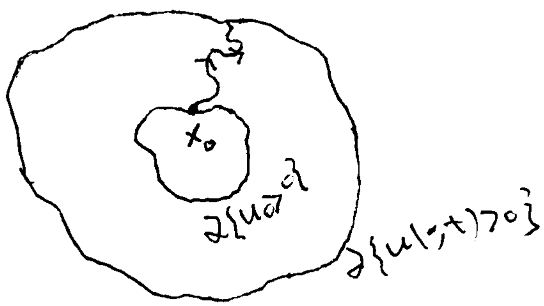
Free Boundary Speed



$u=0$ on $\partial\{u>0\}$

$\Rightarrow \nabla u \cdot \tau = 0$

$\Rightarrow -\nabla u \cdot \nu = |\nabla u|$



Start at x_0,

$x_0 \rightsquigarrow s(t)\nu$, s.t.

$s(t)\nu \in \partial\{u(\cdot,t)>0\}$

$$\frac{d}{dt}u(s(t)\nu,t) = (\nabla u\cdot\nu)\dot{s} + u_t = 0$$

$$\Rightarrow \quad \dot{s} = \frac{-u_t}{\nabla u\cdot\nu} = \frac{u_t}{|\nabla u|} =: V_n$$

I.e. V_n = outward normal velocity.

Classical Free Boundary Sub(super) solution

Develop theory of viscosity solution by comparison with classical soln $\rightsquigarrow$ 1). Comp. principle for visc. soln

2). Existence & Uniqueness

3). Identification with weak soln.

(Caffarelli - Vazquez '99, Brändle - Vazquez '05)

<u>Defn</u>. Nonnegative $u \in C^{2,1}(\overline{\{u>0\}} \times \Sigma)$ is classical free boundary sub(super) soln in Σ if in $\{u>0\} \cap \Sigma$

$$u_t \leq (\geq)\ (m-1)u\Delta u + |Du|^2$$

and on $\Gamma(u) \cap \Sigma$, $|Du| > 0$,

$$V_n \leq (\geq)\ |Du|$$

Viscosity Subsolution

<u>Defn.</u> u is a visc. subsoln if it is continuous and

$\forall \varphi \in C^{2,1}(\Sigma)$ that touches u from above

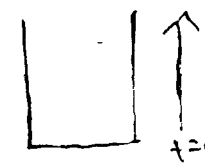
at (x_0, t_0), we have

$$\varphi_t \le (m-1)\varphi \Delta \varphi + |D\varphi|^2$$

"Comparison Principle"

u visc. subsoln, v classical. Then

$u \le v$ on $\partial\Sigma \Rightarrow u \le v$ in Σ.

(

$\partial\Sigma$ = parabolic bdry)

<u>Pf.</u> Approx. v from above:

$$(v_\varepsilon)_t = (m-1)\Delta v_\varepsilon + |D v_\varepsilon|^2 + \varepsilon$$

On $\partial\Sigma$: $v_\varepsilon = v + \varepsilon$

Have $v_\varepsilon \downarrow v$ locally unif. So if $v_\varepsilon = u$ at $P_0 \in \Sigma$,

then $$(v_\varepsilon)_t \le (m-1)\Delta v_\varepsilon + |D v_\varepsilon|^2 \quad *.$$

Viscosity Supersolution

Defn u is a visc. supersoln if it is continuous and

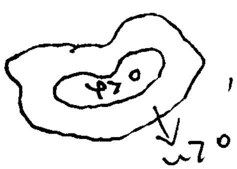
(i). $\forall \varphi \in C^{2,1}(\Sigma)$ that touches u from below at $(x_0, t_0) \in \{u > 0\} \cap \Sigma$

$$\varphi_t \geq (m-1)\varphi \Delta \varphi + |\nabla \varphi|^2$$

(ii). $\forall$ classical free-boundary subsoln φ in Σ:

If $\varphi \prec u$ on $\partial\Sigma$, then $\varphi \leq u$ in Σ.

$\varphi \prec u$:



, $\varphi < u$ inside

I.e.,

not allowed.

Rmk. Get (ii) for visc. subsoln "for free", again by approx. from above and Max. principle...

Viscosity Solution

Defn u is a <u>viscosity solution</u> if it is both a viscosity subsolution and a viscosity supersolution.

Rmk 1). Note there is minimal regularity assumption on u.

2). u is defined to satisfy comparison principle with classical (free boundary) sub(super)solutions.

Istance u^ε solve PME with $u_0^\varepsilon = u_0 + \varepsilon$, then $u^\varepsilon \searrow u^*$ a viscosity soln with initial data u_0.

Note that u^* is also <u>maximal</u> viscosity solution.

To prove uniqueness (i.e, u^* is minimal) requires Comp. principle with <u>visc. solns</u>.

Solutions For Comparison

Barenblatt ((sub)solution)

$$B(x,t,\tau;C)=\frac{\left(C(t+\tau)^{2\lambda}-k|x|^2\right)_+}{(t+\tau)}$$

$$\lambda=\left((m-1)d+2\right)^{-1}$$

$$\lambda = 2k$$

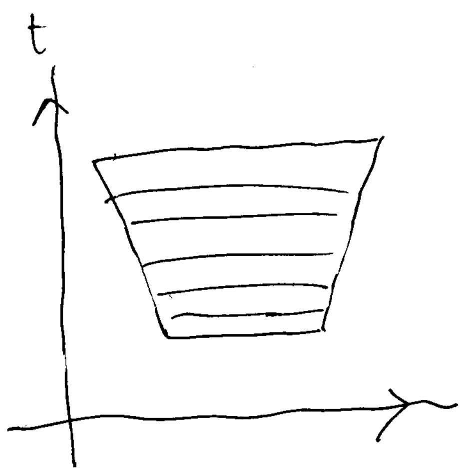
Travelling Wave

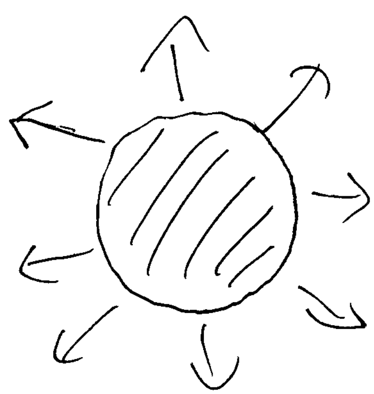
$$H(x,t,A,\omega)=A(|x|+\omega t-B)_+$$

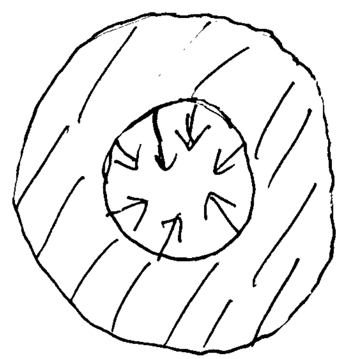
in $\{|x|\le R\}\times[\omega^{-1}(R-B),0]$

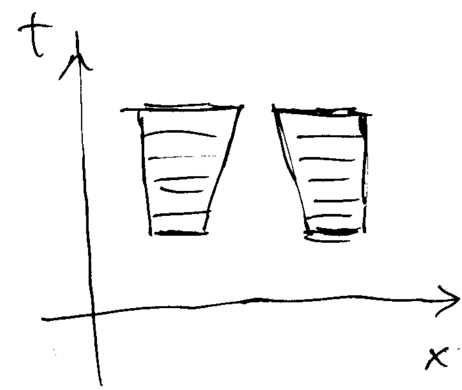
$$R/2<B<R$$

$$\omega/A>1+2(m-1)(d-1)\frac{R-B}{R}$$









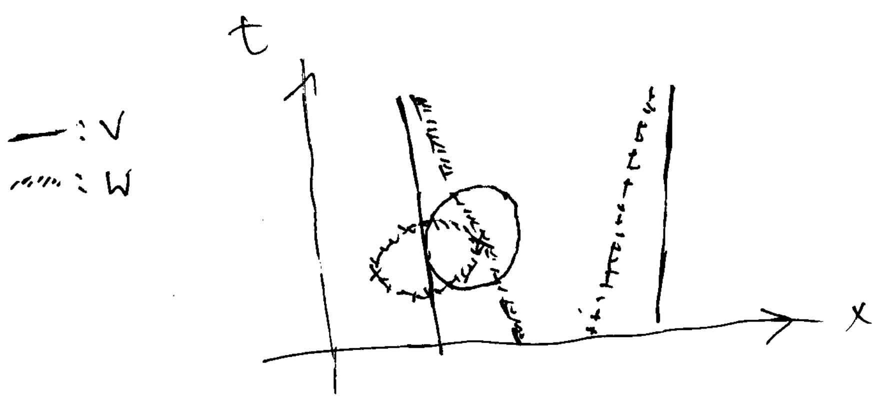
Comparison Principle

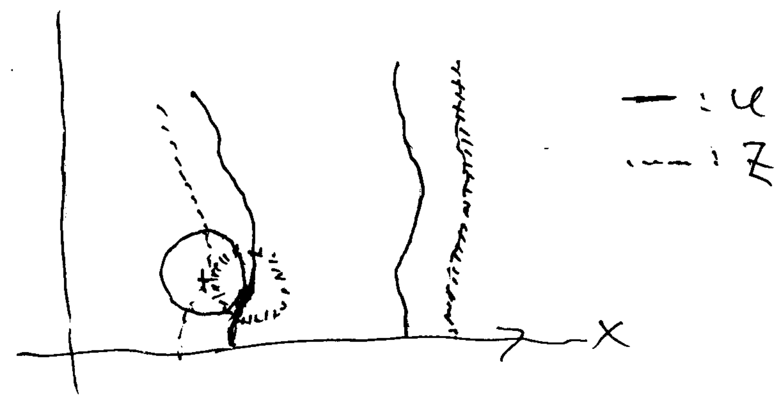
u visc. subsoln, v visc. supersolution.

If $u_0 < v_0$, then $u(x,t) \le v(x,t)$, $\forall (x,t) \in \Sigma$.

Regularite

$$W(x,t) = \inf_{\bar{B}_{r-\delta t}(x,t)} v(y,\tau) \qquad Z(x,t) = \sup_{\bar{B}_r(x,t)} u(y,\tau)$$





Enough to show $Z < W$ always, then $\delta \to 0$, $r \to 0$

No contact in positivity set by Strong Max. Principle.

Also no free boundary contact . . .

Identification with Weak Solution

Weak sol^n. Back in density variable: $\varphi \in C^{2,1}$

$$\int_\Omega \rho(t)\varphi(t) = \int_\Omega \rho(0)\varphi(0) + \int_0^t \int_{\Omega_t} \rho\varphi_t + \rho^m \Delta\varphi$$

Have from e.g., (Bertsch - Hilhorst '86)

1). $u^\varepsilon \searrow u$, $u_0^\varepsilon = u_0 + \varepsilon$, u weak sol^n.

2). uniqueness of weak sol^n.

Comparison principle $\Rightarrow$ uniqueness of viscosity sol^n.

$\Rightarrow$ identification of weak & visc. sol^n.

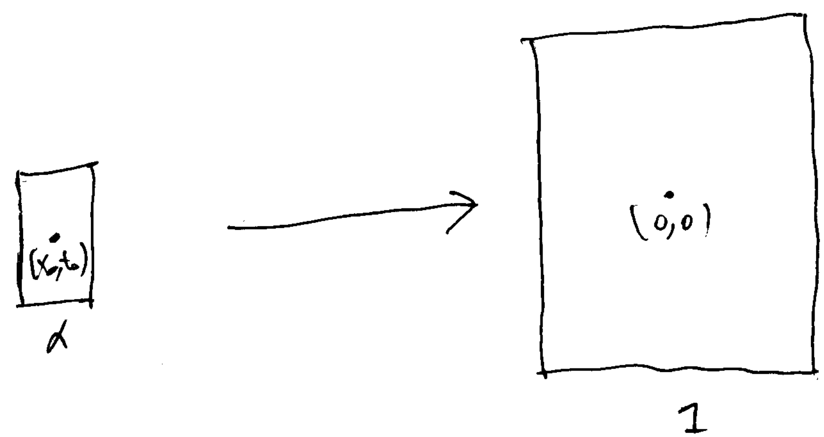
Added Drift Term (joint work w. I. Kim)

$$\rho_t = \Delta(\rho^m) + \nabla\cdot(\rho\nabla\Phi)$$

(PME-D)

$$u_t = (m-1)u\Delta u + |\nabla u|^2 + \nabla u\cdot\nabla\Phi + (m-1)u\Delta\Phi$$

Can do other things, but seems best:
Use hyperbolic scaling to suitably modify
Barenblatt and traveling wave solutions.



$$u(x,t) \longmapsto V_1(x,t) = \alpha^{-1}u(\alpha(x-x_0),\alpha(t-t_0))$$

u satisfies PME-D, then v_1 satisfies ($b = \nabla\Phi(x_0,t_0)$)

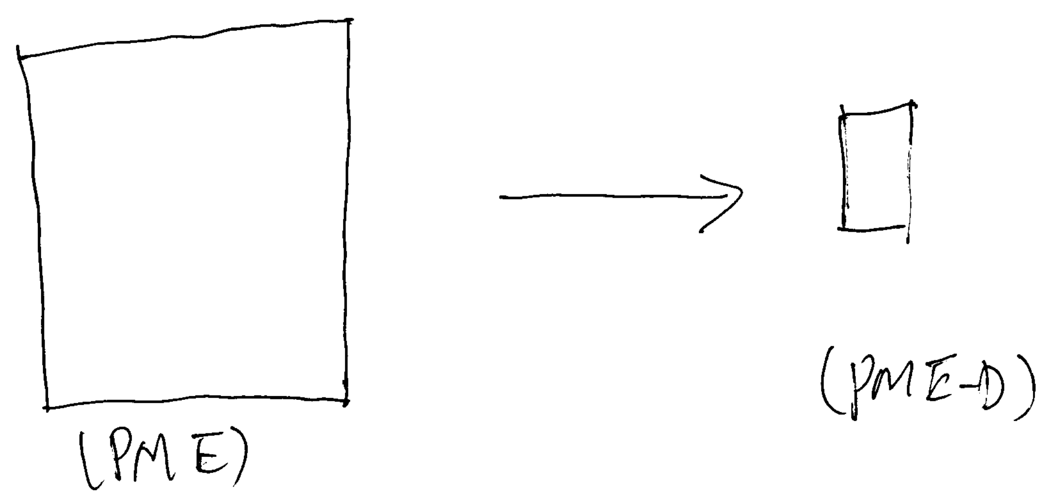
$$(v_1)_t = (m-1)\,v_1 \Delta v_1 + |\nabla v_1|^2 + \vec{b}\cdot\nabla v_1$$
$$+ O(\alpha)\,\nabla v_1 + \alpha(m-1)\,v_1 \Delta\Phi$$

and $v = v_1(x - \vec{b}t, t)$ satisfies

$$v_t = (m-1)\,v\Delta v + |\nabla v|^2 + O(\alpha)\,\nabla v + O(\alpha)\,v.$$

<u>Key point</u>. Φ-dependence in higher order terms.

To build test fcns, reverse:



Have, e.g.,

Lemma. u visc. soln of PME in $B_{1+\alpha}(0)\times[-1,1]$

then $$\underline{u}(x,t) = \inf_{y\in B_{\alpha-\alpha t}(x)} e^{\alpha t} u(y,t)$$

is subsoln of

$$(\underline{u})_t = (m-1)\,\underline{u}\,\Delta\underline{u} + |D\underline{u}|^2 - \alpha|D\underline{u}| - \alpha\underline{u}$$

Hence if $H(x,t;A,w) = A(|x|+wt-B)_+$ supersoln for PME in cylinder of unit scale,

$$\tilde{H}(x,t) = \alpha\,\underline{H}(\alpha^{-1}(x-x_0+\vec{b}(t-t_0)),\alpha^{-1}(t-t_0);C(\alpha))$$

is (classical free boundary) supersoln of PME-D in cylinder of α-scale.

Following general steps as before, modifying defn. of visc. soln. appropriately, and using these test fcns for comparison, we obtain:

<u>Thm.</u> (Kim, L.) $\exists!$ viscosity solution for PME-D. and it coincides with the weak solution.

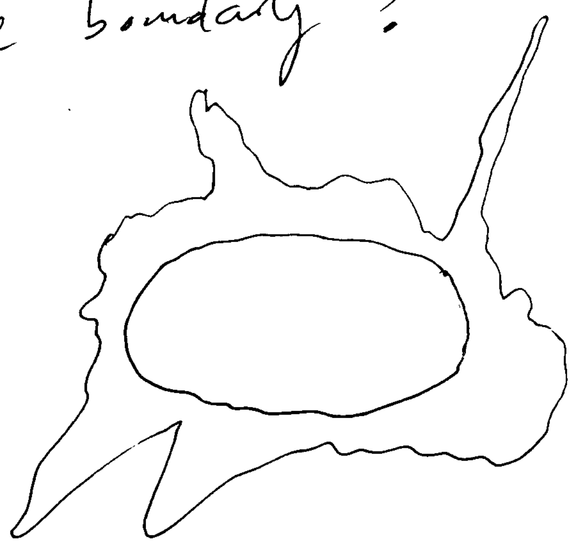
As application, we investigate free boundary convergence as $t \to \infty$ ...

Convergence of Free Boundary

Monotone $\Phi \in C^2$: $|\nabla \Phi| > 0$ except at $x = x_0$ (minimum)

Know $u(\cdot, t) \longrightarrow (\underbrace{C_*}_{\text{determined by mass.}} - \Phi)_+ := u_\infty$ uniformly (BH '86)

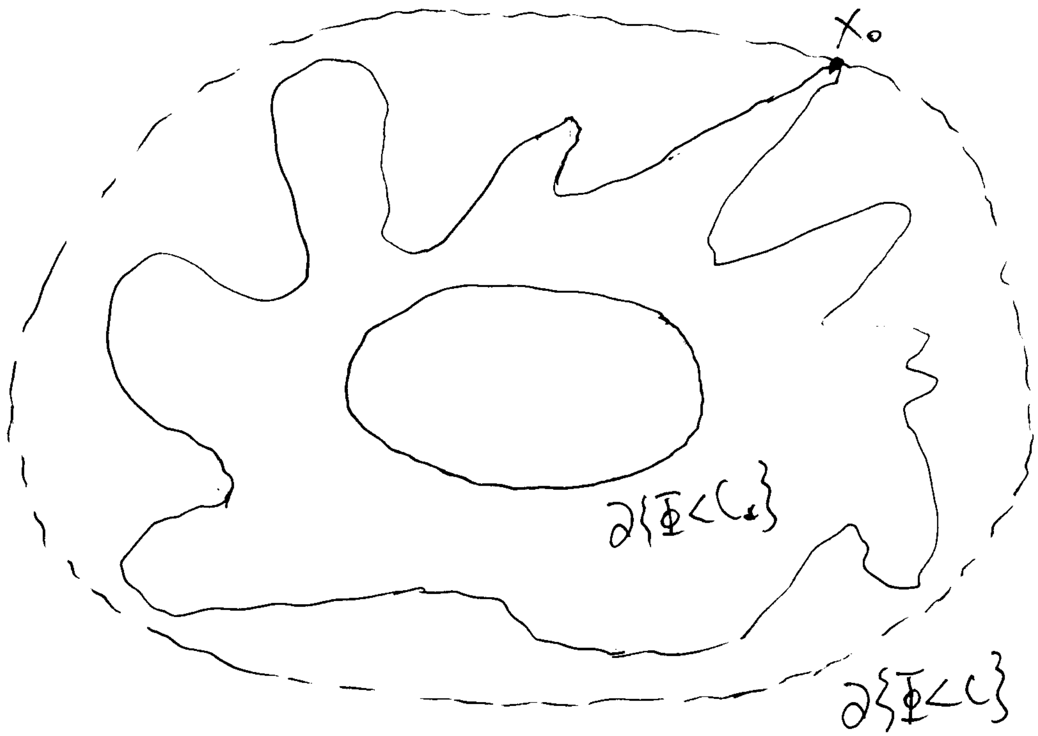
Free boundary?



tentacles as $t \to \infty$?

Have $\Gamma(u_\infty) \supseteq \lim \Gamma_t(u)$ by unif. convergence.

Need to show $\Gamma_t(u) \to \Gamma(u_\infty)$ from outside.



- Fix $T > 0$
- Comp. w/ supersoln $\Rightarrow$ cpct support
- C smallest s.t. $\{\Phi < C\} \supseteq \Omega_T(u) := \{u(\cdot, T) > 0\}$
- $\exists x_0 \in \partial\{\Phi < C\} \cap \overline{\Omega_T}(u)$.

Basically want to show:

At some $T+\varepsilon$, x_0 has moved inwards ...

We have:

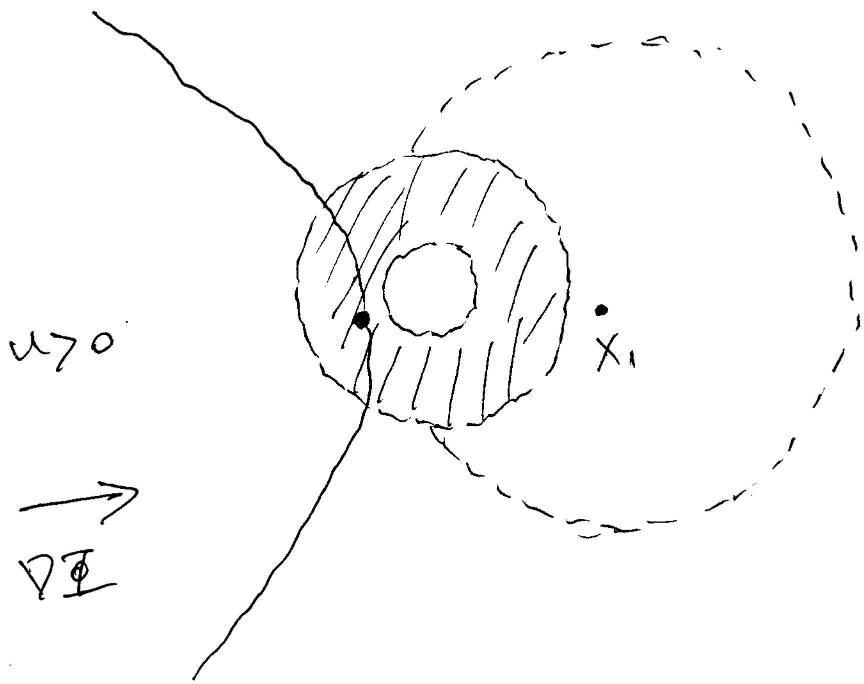
At x_0,
∇u and $\nabla\Phi$
are antiparallel.

Recall $\quad V_n = \dfrac{u_t}{|\nabla u|} = |\nabla u| + \nabla\Phi \cdot \dfrac{\nabla u}{|\nabla u|}$

So formally, at x_0,

$$V_n = |\nabla u| - |\nabla\Phi| < 0 \qquad (|\nabla\Phi| \neq 0)$$

Rigorously,



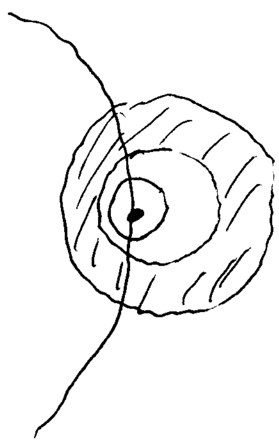
- At time T^*, fit under supersolution $\tilde{H}$
- By construction of $\tilde{H}$,

$$V_n(\tilde{H}) = \underbrace{\omega}_{V_n(H)} + \underbrace{\nabla\Phi\cdot\frac{\nabla H}{|\nabla H|}}_{<0} + \underbrace{C\alpha}_{\text{small}}$$

$$< 0$$

($\nabla\Phi$ pulls $\Gamma_t(u)$ inwards)

Later:



- At time $T^*+\varepsilon$, $u=0$ in $B_\delta(x_0)$

Rate of Convergence

Have L^1-rate of convergence from

Thm ((Carrillo - Jüngel - Markowich - Toscani - Unterreiter, '01)

Suppose Φ is convex. Then $\exists$ constants s.t.

$$\int |u(x,t) - u_\infty(x)|\, dx \le K e^{-\alpha t}$$

To quantify rate of free boundary convergence,

need to convert

L^1-estimate $\rightsquigarrow$ pointwise estimate.

Use comparison + Hölder regularity

for weak solns...

We have

Thm (Kim, L.)

- If Φ is monotone, then $\Gamma_t(u) \longrightarrow \Gamma_\infty(u)$ in the Hausdorff distance.
- If Φ is convex, we have further that for T sufficiently large,

  $\Gamma_T(u)$ is in the $Ke^{-\alpha t}$-nbhd of $\Gamma(u_\infty)$.

  Here K, α depend on m, $\sup u_0$, and Φ.

Thank you